This problem is about Lie group actions on flags. I define a complete flag in an $n$-dimensional vector space $V$ to be a chain of subspaces 

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V, \quad \dim V_j = j.$$ 

The group $GL(V)$ acts transitively on the set of complete flags in $V$. So far this has nothing to do with the underlying field, or with topology, or with Lie groups.

Suppose now that $F = \mathbb{R}$, and that $Q_0$ is a positive-definite quadratic form on $V$. I proved that there is a bijection 

$$\{\text{complete flags in } V\} \leftrightarrow \{n\text{-tuples } (L_1, \ldots, L_n) \text{ of orthogonal lines in } V\}.$$ 

The bijection is defined by 

$$V_j = \text{Span}(L_1, \ldots, L_j), \quad L_j = V_{j-1}^\perp \cap V_j.$$ 

An easy consequence of this bijection is that 

$$\text{complete flags in } V \simeq O(V)/[O(1)^n];$$

this identification (which requires choosing an orthonormal basis of $V$ as a base-point) is a homeomorphism respecting the action of $O(V)$. In particular, this shows that the homogeneous space of all complete flags is compact.

Suppose $W$ is a complex vector space. Recall that a Hermitian form on $W$ is a map 

$$h: W \times W \to \mathbb{C}$$

subject to the requirements 

$$h(av_1 + bv_2, w) = ah(v_1, w) + bh(v_2, w), \quad h(v, w) = h(w, v),$$

which is nondegenerate: for every nonzero $v \in W$ there is a $w \in V$ such that $h(v, w) \neq 0$. The unitary group of $(W, h)$ is 

$$U(W) = \{g \in GL(V) \mid h(g \cdot v, g \cdot w) = h(v, w) \ (v, w \in V)\}.$$ 

Suppose now that $V$ is a real vector space of dimension $2n$. Recall that a symplectic form on $V$ is a skew-symmetric bilinear map 

$$\omega: V \times V \to \mathbb{R}$$

that is nondegenerate: for every nonzero $v \in V$ there is a $w \in V$ such that $\omega(v, w) \neq 0$. The symplectic group of $(V, \omega)$ is 

$$Sp(V) = \{g \in GL(V) \mid \omega(g \cdot v, g \cdot w) = \omega(v, w) \ (v, w \in V)\}.$$ 

If $U$ is a subspace of $V$, define 

$$U^\perp = \{v \in V \mid \omega(u, v) = 0 \text{ all } u \in U\}.$$
Taking \( \perp \) reverses inclusion of subspaces, and \((U^\perp)^\perp = U\). If \( U^\perp \cap U = 0 \), we say that the subspace \( U \) is symplectic. If \( I^\perp \cap I = I \) (the opposite extreme), we say that the subspace \( I \) is isotropic.

A complete symplectic flag in \( V \) is a chain of symplectic subspaces
\[
0 = U_0 \subset U_2 \subset \cdots \subset U_{2n} = V, \quad \dim U_{2j} = 2j.
\]

A complete isotropic flag in \( V \) is a chain of isotropic subspaces
\[
0 = I_0 \subset I_1 \subset \cdots \subset I_n, \quad \dim I_j = j.
\]

Given two symplectic vector spaces \((V, \omega_V)\) and \((W, \omega_W)\), we can define a symplectic structure on \( V \oplus W \) by
\[
\omega_{V \oplus W}((v_1, w_1), (v_2, w_2)) = \omega_V(v_1, v_2) + \omega_W(w_1, w_2).
\]

For the problems, you may use the (easy) fact that if \( U \) is a symplectic subspace of \( V \), then so is \( U^\perp \), and \( V \cong U \oplus U^\perp \) (as symplectic vector spaces).

A symplectic basis of \( V \) is a list of vectors in \( V \)
\[
(e_1, \ldots, e_n, f_1, \ldots, f_n)
\]
subject to the conditions
\[
\omega(e_i, e_j) = \omega(f_i, f_j) = 0, \quad \omega(e_i, f_j) = \delta_{ij}. \quad \text{(SYMP)}
\]

In all problems, \((V, \omega)\) is a symplectic vector space of dimension \( 2n \).

1. a) Prove that \( Sp(V) \) acts in a simply transitive way on symplectic bases of \( V \).
   b) Suppose \((e_1, \ldots, e_m)\) is a linearly independent list in \( V \), and that \( \omega(e_i, e_j) = 0 \).
      Prove that \((e_i)\) can be completed to a symplectic basis of \( V \).
   c) Suppose \((e_1, \ldots, e_m, f_1, \ldots, f_m)\) is a list satisfying conditions like (SYMP) above. Prove that \((e_i, f_i)\) can be completed to a symplectic basis of \( V \).

2. a) Show that \( Sp(V) \) acts transitively on the set of complete symplectic flags in \( V \).
   b) Show that \( Sp(V) \) acts transitively on the set of complete isotropic flags in \( V \).
   3. Prove that there is in \( Sp(V) \) an element \( J \) such that \( J^2 = -I \).

Using \( J \) as in Problem 3, you can make \( V \) into a complex vector space by defining complex scalar multiplication
\[
(x + iy) \cdot v = x \cdot v + y \cdot Jv.
\]

4. Suppose \( J \) is as in problem 3, and \( V \) is regarded as a complex vector space as above. Show that there is a nondegenerate Hermitian form \( h \) so that
\[
\text{Cent}_{Sp(V)}(J) = U(V, h).
\]

5. Suppose \( h \) is positive definite. Prove that \( U(V, h) \) acts transitively on the set of complete isotropic flags in \( V \).

6. Suppose \( h \) is neither positive nor negative definite. Prove that \( U(V, h) \) does not act transitively on the set of complete isotropic flags in \( V \). How many open orbits (of \( U(V, h) \) on complete isotropic flags) are there?