1. Suppose $A$ is an $n \times n$ real matrix. Prove that

$$\exp(A) = \lim_{N \to \infty} (I + \frac{1}{N}A)^N.$$ 

Prove also that

$$\det(\exp(A)) = \exp(\operatorname{tr} A)$$

(with $\operatorname{tr} A$ the sum of the diagonal entries of $A$).

Using the binomial theorem, we compute

$$(I + \frac{1}{N}A)^N = \sum_{p=0}^{N} \binom{N}{p} I^{N-p}(A/N)^p$$

$$= \sum_{p=0}^{N} \left( \frac{N!}{p!(N-p)!} (A/N)^p \right)$$

$$= \sum_{p=0}^{N} \left( \frac{N(N-1) \cdots (N-p+1)}{p!} (A/N)^p \right)$$

$$= \sum_{p=0}^{N} \left( \left[ \frac{N}{N} \cdot \frac{N-1}{N} \cdots \frac{N-p+1}{N} \right] A^p \right)$$

The term in square brackets in the last line is between 0 and 1, a fact that we will use repeatedly below. For a fixed value of $p$, it converges to 1 as $N$ tends to infinity.

For an $n \times n$ matrix $X$ of real numbers, define $\|X\|$ to be $n$ times the largest absolute value of an entry of $X$. Then

$$\|XY\| \leq \|X\|\|Y\|.$$ 

I talked about essentially this in proving that the series for $\exp(B)$ converges absolutely: the conclusion was

$$\left\| \frac{1}{k!} B^k \right\| \leq \frac{\|B\|^k}{k!},$$

so the series for $\exp(B)$ converges “faster” than the power series for $\exp(\|B\|)$.

Given $A$, and $\epsilon > 0$, choose $N_1$ so large that

$$\sum_{p=N_1+1}^{\infty} \frac{\|A\|^p}{p!} < \epsilon/2.$$ 

Then clearly (for $N \geq N_1$)

$$\left\| \sum_{p=N_1+1}^{\infty} \left( 1 - \left[ \frac{N}{N} \cdot \frac{N-1}{N} \cdots \frac{N-p+1}{N} \right] \frac{A^p}{p!} \right) \right\| < \epsilon/2.$$
Finally, choose \( N_0 \geq N_1 \) so large that
\[
\left( 1 - \left[ \frac{N}{N} \cdot \frac{N-1}{N} \cdots \frac{N-p+1}{N} \right] \right) < \frac{\epsilon \exp(-\|A\|)}{2}
\]
for \( 0 \leq p \leq N_1 \). Then for \( N \geq N_0 \), we have
\[
\| \exp(A) - (I + \frac{1}{N} A)^N \| = \| \sum_{p=N_1+1}^{N_1} \left( 1 - \left[ \frac{N}{N} \cdot \frac{N-1}{N} \cdots \frac{N-p+1}{N} \right] \right) \frac{A^p}{p!} + \sum_{p=N_1+1}^{\infty} \left( 1 - \left[ \frac{N}{N} \cdot \frac{N-1}{N} \cdots \frac{N-p+1}{N} \right] \right) \frac{A^p}{p!} \|
\]
\[
\leq \frac{\epsilon \exp(-\|A\|)}{2} \sum_{p=0}^{N_1} \frac{\|A\|^p}{p!} + \epsilon/2
\]
\[
< \epsilon.
\]
Here we use the choice of \( N_0 \) to get the inequality in the first sum and the choice of \( N_1 \) for the second.

Using this limit formula, we deduce
\[
\det(\exp(A)) = \lim_{N \to \infty} \det(I + \frac{1}{N} A).
\]
The function \( \det M \) can be written as a sum of \( n! \) terms, each of which is of the form
\[
\pm m_{1, \sigma(1)} \cdot m_{2, \sigma(2)} \cdots m_{n, \sigma(n)}.
\]
From this it is easy to calculate that
\[
\det(I + \frac{1}{N} A) = 1 + \frac{\text{tr} A}{N} + O(1/N^2).
\]
Here the last term means some function of \( N \) that is less than or equal to \( C/N^2 \) in absolute value for some \( C \) (depending on \( A \) but not on \( N \)). It’s a calculus exercise that
\[
\lim_{N \to \infty} (1 + b/N + O(1/N^2))^N = \exp(b),
\]
giving the formula we want.

2. Define
\[
GL^+(n, \mathbb{R}) = \{ A \in GL(n, \mathbb{R}) \mid \det(A) > 0 \}.
\]
We proved in class that
\[
\exp: \mathfrak{gl}(n, \mathbb{R}) \to GL^+(n, \mathbb{R}).
\]
Is this map surjective? (Remember that \( \mathfrak{gl}(n, \mathbb{R}) \) means all \( n \times n \) real matrices.)

The answer is no for \( n \geq 2 \). I’ll prove this just for \( n = 2 \). If \( B \) is any \( 2 \times 2 \) real matrix, then the two complex eigenvalues of \( B \) (the roots of the characteristic polynomial \( t^2 - (\text{tr} B)t + \det B \)) are either

1. two real numbers \( \beta_1 \) and \( \beta_2 \), or
2. two complex conjugate complex numbers \( a \pm bi \) (with \( b \neq 0 \)).
Therefore the two eigenvalues of \( \exp(B) \) are either

1. two positive real numbers \( \exp(\beta_1) \) and \( \exp(\beta_2) \), or
2. two complex conjugate complex numbers

\[
\exp(a \pm bi) = e^a \exp(\pm bi) = r \exp(\pm bi)
\]

(with \( r = e^a \)).

In the second case, the two complex numbers have the same norm.

In light of this description, we see that the matrix

\[
B = \begin{pmatrix} -2 & 0 \\ 0 & -1/2 \end{pmatrix}, \quad \det(B) = 1
\]

cannot be in the image of \( \exp \). Its two eigenvalues \(-2\) and \(-1/2\) are not positive real, so they do not fall in case (1); but they are not complex conjugates of each other, so they do not fall in case (2).

Describing the image of \( \exp \) precisely is a bit complicated. The matrix

\[
\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}
\]

has the right eigenvalues (a complex conjugate pair) to be in the image, but in fact it is not. On the other hand, the matrix

\[
\begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}
\]

is in the image of \( \exp \).

3. Suppose \( A \) is an \( n \times n \) real matrix. Find necessary and sufficient conditions on \( A \) for the one-parameter group \( \{ \exp(tA) \mid t \in \mathbb{R} \} \) to be closed in \( \text{GL}(n, \mathbb{R}) \).

Here is the answer: \( \exp(\mathbb{R}A) \) is closed if and only if \( A \) is diagonalizable as a complex matrix; and all the nonzero eigenvalues are purely imaginary numbers \( iy_j \); and all the ratios \( y_j/y_k \) (when \( y_k \neq 0 \)) are rational numbers.

The simplest non-closed example is

\[
A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & -\sqrt{2} & 0 \end{pmatrix}.
\]

The one-parameter subgroup is

\[
\left\{ \begin{pmatrix} \cos t & \sin t & 0 & 0 \\ -\sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos(\sqrt{2}t) & \sin(\sqrt{2}t) \\ 0 & 0 & -\sin(\sqrt{2}t) & \cos(\sqrt{2}t) \end{pmatrix} \mid t \in \mathbb{R} \right\}.
\]
Its closure is the two-dimensional torus
\[
\left\{ \begin{pmatrix} \cos \theta_1 & \sin \theta_1 & 0 & 0 \\ -\sin \theta_1 & \cos \theta_1 & 0 & 0 \\ 0 & 0 & \cos \theta_2 & \sin \theta_2 \\ 0 & 0 & -\sin \theta_2 & \cos \theta_2 \end{pmatrix} \mid \theta_1, \theta_2 \in \mathbb{R} \right\}.
\]

The proof requires some detailed understanding of Jordan canonical form for real matrices. I will just quote a useful version of this, without helping you find a reference for exactly this statement.

**Theorem.** Suppose \( A \) is a linear transformation on a finite dimensional real vector space \( V \). Then there is a unique decomposition
\[
A = A_h + A_e + A_n
\]
subject to the requirements

1. the linear transformations \( A_h \), \( A_e \), and \( A_n \) commute with each other;
2. the linear transformation \( A_h \) is diagonalizable with real eigenvalues;
3. the linear transformation \( A_e \) is diagonalizable over \( \mathbb{C} \), with purely imaginary eigenvalues; and
4. the linear transformation \( A_n \) is nilpotent: \( A_n^N = 0 \) for some \( N > 0 \).

The subscripts \( h \), \( e \), and \( n \) stand for “hyperbolic,” “elliptic,” and “nilpotent.”

Suppose \( f \) is a continuous map from \( \mathbb{R} \) to a metric space. The image \( f(\mathbb{R}) \) can fail to be closed only if there is an unbounded sequence of real numbers \( t_i \) such that \( f(t_i) \) converges in the metric space. (You should think carefully about why this is true: the proof is very short, but maybe not obvious.)

So if the image is not closed, then we can found an unbounded sequence \( t_i \) so that \( \exp(t_iA) \) is convergent in \( GL(n, \mathbb{R}) \), and in particular is a bounded sequence of matrices. By passing to a subsequence, we may assume that all \( t_i \) have the same sign. Since matrix inversion is a homeomorphism, \( \exp(-t_iA) \) is also a (convergent and) bounded sequence of matrices. Perhaps replacing the sequence by its negative, we may assume all \( t_i > 0 \) or all \( t_i < 0 \) (whichever we choose we can get).

Now the Jordan decomposition guarantees
\[
\exp(tA) = \exp(tA_h) \exp(tA_e) \exp(tA_n).
\]
In appropriate coordinates the matrix \( A_e \) is block diagonal with blocks
\[
\begin{pmatrix} 0 & y_j \\ -y_j & 0 \end{pmatrix}
\]
(with \( y_j \neq 0 \)) and zeros; so \( \| \exp(tA_e) \| \) is bounded. The power series for \( \exp(tA_n) \) ends after the term \( t^N A_n^N / N! \); so \( \| \exp(tA_n) \| \) has polynomial growth in \( t \).

If \( A_h \) has a positive eigenvalue, then \( \exp(tA_h) \) grows exponentially in \( t \), so the sequence \( \exp(t_iA) \) cannot be bounded. Similarly, if \( A_h \) has a negative eigenvalue, then \( \exp(-t_iA) \) grows exponentially. The conclusion is that if the image is not closed, then \( A_h = 0 \).

In exactly the same way, suppose \( A_n^N \neq 0 \) but \( A_n^{N+1} = 0 \). Then \( \exp(tA_n) \) grows like a polynomial of degree exactly \( N \); so (because of the boundedness of \( \exp(tA_e) \))
we conclude that \( \exp(tA) \) also grows like a polynomial of degree exactly \( N \). The conclusion is that if the image is not closed, then \( N = 0 \), which means \( A_n = 0 \).

We have shown that the image can fail to be closed only if \( A = A_e \). In this case the image is bounded; so it is closed if and only if it is compact. Suppose that the eigenvalues of \( A = A_e \) are \( iy_j \) as above, so that \( \exp(tA) \) has diagonal blocks

\[
\begin{pmatrix}
\cos(ty_j) & \sin(ty_j) \\
-\sin(ty_j) & \cos(ty_j)
\end{pmatrix}.
\]

If all the ratios \( y_j/y_1 = p_j/q_j \) are rational, then it’s easy to see that \( \exp(tA) \) is periodic with period (dividing)

\[(\text{least common multiple of all } q_j)(2\pi/y_1);\]

so the image is a circle (or a point), and is closed.

Conversely, suppose that the image is compact. If \( y_1 \) and \( y_2 \) are nonzero, then the group

\[
\begin{pmatrix}
\begin{pmatrix}
\cos(ty_1) & \sin(ty_1) \\
-\sin(ty_1) & \cos(ty_1)
\end{pmatrix} & 0 \\
0 & \begin{pmatrix}
\cos(ty_2) & \sin(ty_2) \\
-\sin(ty_2) & \cos(ty_2)
\end{pmatrix}
\end{pmatrix}
\]

is compact if and only if \( y_1/y_2 \) is rational. (I’m tired of typing, so I won’t write out a proof.) By projecting the (assumed compact) \( \{\exp(tA)\} \) on various collections of four coordinates, and using “continuous image of compact is compact,” we deduce that all the ratios \( y_j/y_k \) are rational, as we wished to show.