18.755 second problems, due in class 9/19/17

1. Suppose $A$ is an $n \times n$ real matrix. Prove that

$$\exp(A) = \lim_{n \to \infty} (I + \frac{1}{n}A)^n.$$ 

Prove also that

$$\det(\exp(A)) = \exp(\text{tr } A)$$

(with $\text{tr } A$ the sum of the diagonal entries of $A$).

2. Define

$$GL^+(n, \mathbb{R}) = \{ A \in GL(n, \mathbb{R}) \mid \det(A) > 0 \}.$$ 

We proved in class that

$$\exp: \mathfrak{gl}(n, \mathbb{R}) \to GL^+(n, \mathbb{R}).$$

Is this map surjective? (Remember that $\mathfrak{gl}(n, \mathbb{R})$ means all $n \times n$ real matrices.)

3. Suppose $A$ is an $n \times n$ real matrix. Find necessary and sufficient conditions on $A$ for the one-parameter group $\{ \exp(tA) \mid t \in \mathbb{R} \}$ to be closed in $GL(n, \mathbb{R})$.