18.034 Problem Set 6

Due Monday, April 10 in class.

1. Consider the two vectors

\[ v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \]

and the two matrices

\[ M_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \]

a) Prove that \( v_1 \) and \( v_2 \) are both eigenvectors of both \( M_1 \) and \( M_2 \). What are the eigenvalues?

b) Suppose \( a \) and \( b \) are any two real numbers. Write down a matrix \( M \) with the property that \( v_1 \) and \( v_2 \) are both eigenvectors of \( M \), with eigenvalues \( a \) and \( b \) respectively.

c) Write down a system of two (constant coefficient first-order linear) differential equations with the property that the two sets of functions

\[ x(t) = e^{at}, \quad y(t) = e^{at} \]

and

\[ x(t) = e^{bt}, \quad y(t) = -e^{bt} \]

are both solutions.

d) For the differential equations you wrote in part (c), find explicit formulas for the solutions \( x(t) \) and \( y(t) \) satisfying the initial conditions

\[ x(0) = x_0, \quad y(0) = y_0. \]

2. This problem concerns the appearance of solution curves for the differential equations you wrote in 1(d). A “solution curve” means the path traced in the \((x, y)\) plane by a solution \((x(t), y(t))\) as \( t \) varies from \(-\infty\) to \( \infty\).

a) As long as \( a \) and \( b \) are both non-zero, prove that the four half-lines

\[ \{ sv_1 \mid s > 0 \}, \quad \{ sv_1 \mid s < 0 \}, \quad \{ sv_2 \mid s > 0 \}, \quad \{ sv_2 \mid s < 0 \} \]

are all solution curves.

b) Suppose \( a = 2 \) and \( b = 1 \). In a picture showing the two lines \( y = x \) and \( y = -x \), draw several more solution curves of the equations in 1(d). (Hint: the solution curves off these two lines are parabolas.)

c) Same question as (b), but now with \( a = 1 \) and \( b = 2 \).
**d)** Same question as (b), but now with $a = 1$ and $b = -1$. What is the geometric shape of the solution curves now? What happens to the solutions as $t \to +\infty$? What happens as $t \to -\infty$?

**3.** You have a population of lab animals (varying with time $t$) consisting of $x(t)$ juvenilles, $y(t)$ adults, and $z(t)$ seniors.

The adults reproduce at rate $b$, which contributes a term $by(t)$ to the growth of the juvenile population.

The juvenilles turn into adults at a rate $a$, contributing a term $-ax(t)$ to the growth of the juvenile population, and a term $+ax(t)$ to the growth of the adult population.

The adults turn into seniors at a rate $c$, contributing $-cy(t)$ to the growth of the adult population and $+cy(t)$ to the growth of the senior population.

Finally, the seniors die at a rate $d$, contributing $-dz(t)$ to the growth of the senior population.

Assume that all four constants $a$, $b$, $c$, and $d$ are strictly positive.

**a)** Write a system of three differential equations for the three functions $x(t)$, $y(t)$, and $z(t)$ reflecting the conditions described above.

**b)** Show that $-d$ is an eigenvalue of the system, and write down the corresponding solution of the differential equations.

**c)** Suppose that $b = c$. Show that $0$ is an eigenvalue of the system, and write down the corresponding solution. Is this solution biologically reasonable?

**d)** Still supposing $b = c$, show that $-(a+b)$ is another eigenvalue, and write down the corresponding solution of the differential equations. Is this solution biologically reasonable?

**e)** Assume finally that $b > c$. What can you say about the last two eigenvalues of the system (other than $-d$)?