primes - infinitely many primes. \( / \equiv a(q) \) for \( q \) prime. \( \gcd(a, q) = 1 \).

Riemann Zeta function

\[
\xi(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.
\]

Convergent for \( \Re(s) > 1 \).

(assume \( s \in \mathbb{IR} \) for now)

Use any convergence test, e.g. integral test.

Another representation:

\[
\xi(s) = \prod_p \left(1 - p^{-s}\right)^{-1}
\]

\[
= \prod_p \left(1 + p^{-s} + p^{-2s} + \ldots\right)
\]

(show this expression is convergent: Take partial products over primes \( p \in \mathbb{N} \). Then take limit as \( N \to \infty \))

Follows from unique fact. into primes that two representations are equal.

\[
\log(\xi(s)) = \sum \sum \frac{m^{-1}}{p^m} \cdot p^{-ms}
\]

since \(-\log(1-x) = \sum_{m=1}^{\infty} \frac{x^m}{m}\).

Now for \( m > 2 \), have

\[
\sum_{p} \sum_{m>2} m^{-1} \cdot p^{-ms} < \sum_{p} \sum_{m>2} p^{-m} = \sum_{p} \frac{1}{p \cdot (p-1)} < 1.
\]

Provided \( s > 1 \).

Hence

\[
\log(\xi(s)) = \sum \frac{p^{-s} + c}{p} \quad (c < 1)
\]

\( \Rightarrow \) since \( \lim_{s \to 1^+} \), LHS \( \to 0 \), RHS \( \to 0 \).
\[ \sum_{p} p^{-s} \rightarrow \infty \text{ as } s \rightarrow 1^+, \text{ i.e. } \sum \frac{1}{p} \text{ diverges.} \]

Dirichlet. mimic this proof for primes \( p \equiv a (q) \), \( q \) prime.

Somehow end up with \( \sum \frac{1}{p} \) on RHS, LHS: divergent function, prove divergent using \( \zeta(s) \).

Idea: use characters mod \( q \).

(Seen examples of characters mod \( q \). Power residue symbols, trivial character.)

Stated that they form a group under mult. Investigate further.

Pick primitive root \( g \ (mod q) \). Then any \( n \ (mod q) \) is power of \( g \): \( g^{v(n)} = n (q) \) \( v(n) \): index of \( n \).

( Depends on choice of primitive roots )

e.g. \( mod 7 \): \( \phi(6) = 2 \) prim. roots, \( 3, 5 \)

\[ \begin{align*}
3^2 & \equiv 2 \ (7) \quad v_3(2) = 2 \\
5^4 & \equiv 2 \ (7) \quad v_5(2) = 4.
\end{align*} \]

Then given fixed choice of prim. root \( g \). Take complex \( (q-1)^{st} \) rt. of unity: \( \omega \)

Define: \( \chi(n) = \omega^{v(n)} \) (or better: \( \omega^{v(g(n))} \))

( This for \( (n,q) = 1 \). Sometimes extend to all \( n \in \mathbb{Z} \)

with \( \chi(n) = 0 \) if \( q \mid n \). )
Note: \( \omega \) need not be primitive \((q-1)\)st rt. of unity.

Any choice of \( \omega \) gives a character.

\( \omega = (-1) \) gives Legendre symbol. \( \omega = 1 \) gives trivial character.

so have \( q-1 \) different characters mod \( q \).

(note they are characters since if \( n \equiv n_1 n_2 \pmod{q} \) then
\[
\chi(n) \equiv \chi(n_1) + \chi(n_2) \pmod{q-1}
\]

i.e. \( \omega \chi(n) = \omega \chi(n_1) + \omega \chi(n_2) = \omega \chi(n_1) \cdot \omega \chi(n_2) \)

so \( \chi(n) = \chi(n_1) \cdot \chi(n_2) \) as desired.

Do we get (yet more) characters by choosing different primitive roots?

no. e.g. \( \omega = 8_6 \) \( \chi(n) = (8_6)^{3 \chi(n)} \)

Can find on \( 8_i \) s.t. \( \chi(n) = (8_i)^{5 \chi(n)} ? \) HW.

\[
\text{Key property: } \sum_{\chi} \chi(n) = 0 \text{ if } n \not\equiv 0 \pmod{q}.
\]

Idea \( \sum \chi(n) = \sum_{\omega} \omega \chi(n) \). But know \( \sum \omega^k \), for any \( k \),

\[
\omega \text{ so } \begin{cases} 
0 & \text{if } k \not\equiv 0 \pmod{q-1} \\
q-1 & \text{if } k \equiv 0 \pmod{q-1}
\end{cases}
\]

Sneaky idea: Consider
\[
\sum_{\chi} \overline{\chi}(a) \cdot \chi(n) = \sum_{\omega} -\omega^a \cdot \omega \chi(n) = \begin{cases} 
0 & \text{if } n \equiv a \pmod{q} \\
q-1 & \text{if } n \not\equiv a \pmod{q}
\end{cases}
\]
Sketch of Dirichlet’s pf.

\[ L_\omega(s) = \sum_{n=1}^{\infty} \frac{\omega(n)}{n^s} = \sum_{n=1}^{\infty} \frac{X(n)}{n^s} \]

\[ |X(n)| = 1 \text{, so conv. for } \Re(s) > 1. \text{ Moreover, } \omega(n) \text{ is a multiplicative function, so we may write:} \]

\[ L_\omega(s) = \prod_{\text{prime } p} \left( 1 - \omega(p)p^{-s} \right)^{-1}, \text{ for } s > 1. \]

(\text{check that } |\omega(p)p^{-s}| = p^{-s} < \frac{1}{2} \text{ when } s > 1 \text{ so no terms in prod. are 0, hence } L_\omega(s) \neq 0 \text{ for } s > 1). \]

Take logs again:

\[ \log L_\omega(s) = \sum_{p \neq q} \sum_{m=1}^{\infty} \frac{\omega(p^m)}{p^ms} \]

Consider:

\[ \frac{1}{q-1} \cdot \sum_{w} \omega_{w(a)} \cdot \log L_\omega(s) \]

\[ = \sum_{p} \sum_{m=1}^{\infty} \frac{p^{-ms}}{p^m \equiv a (q)} \]

estimate away terms with \( m > 2 \). Leaves sum we want.
Just need to show:

\[ \sum_{q=1}^{+\infty} \omega^{-v(q)} \log L_{\omega}(s) \to \infty \text{ as } s \to 1^+. \]

This will prove \( \sum_{p=a(q)}^{\infty} \frac{1}{p} \) divergent.

On LHS: taking \( \omega = 1 \) gives \( \log L_{1}(s) \) where

\[ L_{1}(s) = \prod_{n=1}^{\infty} \frac{1}{1 - p^{-s}} = \prod_{\text{primes } \ell} (1 - \ell^{-s}) \cdot \zeta(s). \]

So if \( \zeta(s) \to \infty \text{ as } s \to 1^+ \), \( L_{1}(s) \to \infty \text{ as } s \to 1^+ \).

\( 1 - \ell^{-s} \to 1 - \frac{1}{\ell} \)

Remains to show: \( \lim_{s \to 1^+} \log L_{\omega}(s) \) doesn't screw this up. (i.e. is bounded as \( s \to 1^+ \))

Use a little analysis to reformulate this question:

Claim: \( L_{\omega}(s) \), \( \omega \neq 1 \), is convergent for \( s > 0 \) (not just \( s > 1 \)).

Use Dirichlet's test for convergence: \( \sum_{n=1}^{m} a_{n} b_{n} \) bounded, not ind. terms.

Given \( \sum_{n=1}^{\infty} a_{n} b_{n} \) bounded. \( \sum_{n=1}^{\infty} a_{n} \) decreasing, limit 0.

then \( \sum_{n=1}^{\infty} a_{n} b_{n} \) converges.

Let \( b_{n} = n^{-s} \), \( a_{n} = \omega^{v(n)} \). Note that sums \( \sum_{n=1}^{\infty} d_{n} \)

are bounded since the sum over any \( q \) consecutive integers = 0.

(complete residue class)

In fact, uniformly convergent \( \omega \) r.t. \( s \)

for any \( s > \delta > 0 \) (bounded away from 0). So

enough to show \( L_{\omega}(1) \neq 0 \).
Cases: \( \omega \) not real \( (\omega \neq 1, -1) \), \( \omega \) real \( (\omega = -1) \).

Suppose \( \omega \) complex.

Set \( a = 1 \) \& in our earlier equation:

\[
\sum_{q=1}^{\infty} \sum_{m=1}^{q} \frac{1}{m} \log(L_\omega(s)) = \sum_{p=1}^{\infty} \sum_{m=1}^{p} \frac{1}{m} \log(p^{-m}s^{-1})
\]

RHS has all positive terms. \( \Rightarrow \sum_{\omega} \log(L_\omega(s)) > 0 \).

I.e. \( \prod_{\omega} L_\omega(s) \geq 1 \), for any \( s > 1 \).

If \( \exists \omega \) (not real) with \( L_\omega(1) = 0 \), then \( L_{\overline{\omega}}(1) = 0 \), with \( \overline{\omega} \): complex conj. (since, for \( s \) real, \( L_{\overline{\omega}}(1) = \overline{L_\omega(1)} \)).

Conclusion: 2 factors in \( \prod_{\omega} \) have limit 0 as \( s \rightarrow 1^+ \).

1 factor, \( L_1(s) \), has limit 0 as \( s \rightarrow 1^+ \).

Other factors bounded, so could contribute 0.

Idea: 2+ factors of \( \prod_{\omega} \) with limit 0 will win out over \( L_1(s) \) with \( \frac{\text{left side}}{\text{right side}} \) limit 0.

Giving contradiction to fact that \( \prod_{\omega} L_\omega(s) \geq 1 \) for any \( s > 1 \). (taking limit of both sides)

Need to analyze behavior at \( s = 1 \) further:

Want \( L_1(s) < \frac{c}{s-1} \), some constant.

Know \( L_1(s) = \frac{1}{1-q^{-s}} \sum_{n=1}^{\infty} \frac{1}{n^s} < \frac{1}{1-q^{-2}} \sum_{n=1}^{\infty} \frac{1}{n^s} \), and

for \( s \in (1, 2) \)

\[
\sum_{n=1}^{\infty} \frac{1}{n^s} < 1 + \int_{1}^{\infty} \frac{1}{x^s} \, dx = \frac{s}{s-1}.
\]
so take \( c = 2 \cdot (1 - \frac{1}{q^2}) \)
bounded \( L_1(s) \) in range
\( 1 \leq s < 2 \).

For \( L_\omega(s) \), show \( |L_\omega(s)| < \frac{c_2}{s-1} \) \((\text{same for } \overline{\omega})\)

mean value thm: \( \exists \ s_1 \in (1, s) \) with
\[
L_\omega(s) - L_\omega(1) = \frac{L'_\omega(s_1)}{s-1}.
\]
so if \( L_\omega(1) = 0 \), we have \( L_\omega(s) = (s-1) L'_\omega(s_1) \)
Suffices to show \( |L'_\omega(s_1)| \) bounded to get our claim.

But by similar methods as before,
\[
L'_\omega(s) = - \sum_{n=1}^{\infty} \omega(n) \cdot (\log n) n^{-s} \quad \text{again unif. conv. for } s \geq \delta > 0
\]
by Dirichlet's test since
\[
\log n / n^{s} \text{ decreasing for } n \text{ suff. large with } \lim s = 0.
\]

\( \Rightarrow L'_\omega(s) \text{ continuous, for } s > 0 \).
\( \Rightarrow |L'_\omega(s)| \text{ bounded} \).

Putting these into our product \( \prod_{\omega} L_\omega(s) > 1 \), taking limits,
gives contradiction. \((\text{LHS} = 0 \text{ in abs. value}).\)
if $\omega$ real, i.e. $\omega = -1$, then 
\[ L(s_1-1) = \sum_{n=1}^{\infty} \frac{(n)}{\eta^n} \]

where $(\frac{n}{\eta})$ is the Legendre symbol. (Call this $L(s)$ for simplicity)

We must show $L(1) \neq 0$. (Know $L(1) > 0$ since $L(s)$ continuous @ $s=1$
and Euler product shows $L(s) > 0$ for $s > 1$.)

**Plan:** use Gauss sums to express $(\frac{n}{\eta})$, then have $L(s)$ as double sum.
interchange orders of summation.

Recall that $g(n, q) \overset{def}{=} \sum_{m=1}^{q-1} (\frac{m}{q}) e^{2\pi imn/q}$

and 

$g(n, q) = (\frac{n}{q}) \cdot g(0, q)$.

i.e. 

$(\frac{n}{q}) = \frac{1}{g(1, q)} \sum_{m=1}^{q-1} (\frac{m}{q}) e^{2\pi imn/q}$. Substitute into $(*)$

[Seems like no advantage here, but recall Gauss determined exact value of
this sum. (We showed $|g(1, q)| = \sqrt{q}$, at least $\neq 0$ so expression
well-defined.)]

Interchanging orders of summation:

\[ L(1) = \frac{1}{g(1, q)} \sum_{m=1}^{q-1} (\frac{m}{q}) \sum_{n=1}^{\infty} \frac{1}{n} e^{2\pi imn/q} \]

Remember 

$-\log (1-x) = \sum_{n=1}^{\infty} \frac{1}{n} x^n$. Makes sense as complex series
as well. (radius of conv. $< 1$
as real series
conv. @ -1, div. @ 1)

As complex series, conv. for any $z$, $|z| \leq 1$
with $z \neq 1$.

so 

$L(1) = \frac{-1}{g(1, q)} \sum_{m=1}^{q-1} (\frac{m}{q}) \left\{ \log \left| 1 - e^{2\pi im/\eta} \right| + i \left( \frac{\pi m}{\eta} - \frac{\pi}{2} \right) \right\}$
Answer to exact formula for \( g(1,q) = \{ \begin{align*} \frac{q}{b^{1/2}} & \quad \text{if } q = 1, \\ i \frac{q^{1/2}}{b} & \quad \text{if } q = 3. \end{align*} \) (4)

if \( q = 3 \) (4), easier since know \( L(1) \) is real. (all terms real)

so have \( \begin{align*}
L(1) &= -\frac{1}{i q^{1/2}} \sum_{m=1}^{q-1} \left( \frac{m}{q} \right) \cdot \left( i \left( \frac{\pi m}{q} - \frac{\pi}{2} \right) \right) \\
&= -\frac{\pi}{q^{3/2}} \sum_{m=1}^{q-1} \left( \frac{m}{q} \right) \cdot m + c \cdot \sum_{m=1}^{q-1} \left( \frac{m}{q} \right) \\
&= \frac{3\pi}{23^{1/2}} \\
\end{align*} \)

(\text{even w/o knowing } L(1) \text{ real, can see that } m, q-m \text{ terms in sum cancel the } \log(\sin) \text{ factors})

E.g.: \( q = 23 \)

then \( \sum_{m=1}^{23-1} m \cdot \left( \frac{m}{23} \right) = 1 + 2 + 3 + 4 - 5 + 6 - 7 + 8 \cdots - 21 - 22 = -69 \)

so \( L(1) = \frac{3\pi}{(23)^{1/2}} \)

Cool pf. note: \( ^{\wedge} \) has same parity as \( \sum_{m=1}^{q-1} m = q \cdot \frac{(q-1)}{2} \)

\( \sum_{m=1}^{q-1} m \cdot \left( \frac{m}{q} \right) \)

But \( q \cdot \frac{(q-1)}{2} \) is odd since \( q \) odd, so finite sum can't = 0. //

No elementary pf that \( \sum_{m=1}^{q-1} m \cdot \left( \frac{m}{q} \right) \) is always \( < 0 \). (Know true since \( L(s) > 0 \) for \( s > 1 \) and hence \( L(1) > 0 \) )