RESEARCH STATEMENT

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My primary field of interest is geometric analysis, with a focus on minimal surfaces in $\mathbb{R}^3$. Minimal surfaces have broad appeal both within and beyond the mathematical community. As physical objects – occurring naturally as soap films – they can be well imagined and understood by the non-mathematician. Further, with characterizations based in submanifold geometry, partial differential equations, calculus of variations, and complex analysis, minimal surfaces are particularly interesting and accessible to a broad audience of mathematicians.

1. Minimal Surfaces

Let $\Sigma \subset \mathbb{R}^3$ denote a smooth, oriented, embedded surface in $\mathbb{R}^3$, possibly with boundary. Let $\phi \in C_0^\infty(\Sigma)$ and $\eta \in T\Sigma$ be a smooth unit normal vector field. Define $\Sigma_{t,\phi} := \{x + t\phi(x)\eta(x) | x \in \Sigma\}$, a smooth variation of $\Sigma$ with compact support. Define $A(t) = Area(\Sigma_{t,\phi})$. Then, by the first variation formula for area,

$$\frac{dA}{dt}(0) = -\int_\Sigma \phi \cdot H$$

where $H$ denotes the mean curvature of $\Sigma$. That is, $2H = (\kappa_1 + \kappa_2)$ where $\kappa_1, \kappa_2$ are the principle curvatures of $\Sigma$. A surface $\Sigma$ is a minimal surface if it is a critical point for the area functional. Thus (1.1) implies $\Sigma$ is minimal if and only if the mean curvature vanishes identically. Classic examples of minimal surfaces embedded in $\mathbb{R}^3$ are the plane, the helicoid, and the catenoid.

Many recent results and current work depend upon the geometric quantity called the flux. For a closed curve $\gamma \subset \Sigma$ with unit tangent vector $\dot{\gamma}$, we define – after fixing an orientation on $\Sigma$–

$$Flux(\gamma) = \int_\gamma \dot{\gamma} \wedge \eta \in \mathbb{R}^3$$

where $\dot{\gamma} \wedge \eta = \nu$ is the conormal vector field to $\gamma$. As $\nu \cdot e_i = \nu \cdot \nabla_\Sigma x_i$ and $\Delta_\Sigma x = -H \eta = 0$, the divergence theorem implies that the flux of a curve depends only on its homology class.

2. Minimal Surfaces with One End

This section describes the work done in [1, 3].

In the past twenty years, the understanding of complete, embedded minimal surfaces with finite topology – homeomorphic to a finitely punctured compact Riemann surface – has advanced greatly. By a result of Colding and Minicozzi [14], these surfaces are properly embedded (see Section 5). Of continued particular interest is the study of such surfaces with the added condition of having exactly one end.

Throughout this section, let $\mathcal{E}(1)$ denote the set of all complete, embedded minimal surfaces in $\mathbb{R}^3$ with finite genus and one end. The construction in [39], by Hoffman, Weber, and Wolf, confirmed the space contains a surface of positive genus. As is often the case, questions concerning the uniqueness of the embedded example and existence of higher genus surfaces are now a high priority in the field. In attempting to answer these questions, we determine a priori geometric properties of any such surface in the family.
2.1. Conformal Type and Asymptotic Geometry. Meeks and Rosenberg [24] proved the helicoid is the unique non-flat disk in \( \mathcal{E}(1) \) by first establishing any such \( \Sigma \) was conformal to \( \mathbb{C} \) and then examining the Weierstrass data. Along with Bernstein, in [1] I prove the conformal type and asymptotic behavior of the Weierstrass data for all surfaces in \( \mathcal{E}(1) \).

**Theorem 2.1.** Any \( \Sigma \in \mathcal{E}(1) \) is conformally a punctured, compact Riemann surface. Moreover, if the surface is not flat, then, after a rotation of \( \mathbb{R}^3 \), the height differential, \( dh \), extends meromorphically over the puncture with a double pole, as does the meromorphic one form \( \frac{dz}{g} \). (Here \( g \) is the stereographic projection of the Gauss map.)

Both our proof and the one in [24] rely heavily on the lamination theory of Colding and Minicozzi, developed in [10, 11, 12, 13]. Let \( \Gamma \) be an appropriate neighborhood of the puncture, and define \( z = x_3 + ix_3^* \) where \( x_3^* \) is the harmonic conjugate of \( x_3 \). For \( \Sigma \in \mathcal{E}(1) \), Theorem 2.1, the Weierstrass representation, and embeddedness imply that near the puncture the Weierstrass data is asymptotic to that of a helicoid.\(^1\)

**Corollary 2.2.** There exists an \( \alpha \in \mathbb{R} \) such that, on \( \Gamma \), \( g(p) = \exp(i\alpha z(p) + F(p)) \) where \( F : \Gamma \rightarrow \mathbb{C} \) extends holomorphically over the puncture with a zero there.

Using Theorem 2.1, Corollary 2.2, and an earlier result of Hauswirth, Perez, and Romon in [19] – where they studied annular ends of minimal surfaces with conditions on the Weierstrass data – allow us to conclude:

**Corollary 2.3.** If \( \Sigma \in \mathcal{E}(1) \) is non-flat then \( \Sigma \) is \( C^0 \)-asymptotic to some helicoid\(^2\).

These results complete the understanding of the conformal type and asymptotic geometry of minimal surfaces with finite topology.

2.2. Symmetries of Genus One Helicoids. All known embedded genus one helicoids, see [21] and [39], have maximal symmetry – one orientation preserving, and two orientation reversing rotations. In [6], Bobenko conjectures that any immersed genus-\( k \) helicoid (i.e. a minimally immersed, once punctured genus-\( k \) surface with “helicoid-like” behavior at the puncture) is symmetric with respect to rotation by 180° around a line perpendicular to the surface. This conjecture is motivated by the observation in [6] that the period problem for these surfaces is algebraically “well-posed” when there is such a symmetry, but is over-determined without it.

In [3], Bernstein and I prove the conjecture for embedded genus-1 helicoids, and for higher genus with an assumption on conformal type.

**Theorem 2.5.** Let \( \Sigma \in \mathcal{E}(1) \) have genus equal to one. Then there is a line \( \ell \) normal to \( \Sigma \) so that rotation by 180° about \( \ell \) acts as an orientation preserving isometry on \( \Sigma \). Further, if \( \Sigma \in \mathcal{E}(1) \) with genus greater than one, the same result holds with the added assumption that \( \Sigma \) is hyperelliptic.

The proof relies on the interaction between a certain holomorphic involution on the conformal structure of \( \Sigma \) and the so called “period conditions” of the Weierstrass data. The non-existence of an isometry implies \( \Sigma \) has vertical flux, which allows us to use the Lopez-Ros deformation [33] to

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1The helicoid has Weierstrass data \( g = e^{i\alpha z}, dh = dz \) for \( z \in \mathbb{C}, \alpha \in \mathbb{R} \); thus \( \frac{dz}{g} - i \alpha dh \) is identically zero.

2i.e. for any \( \epsilon > 0 \) there exists \( R_\epsilon > 0 \) so that the part of the end outside of \( B_{R_\epsilon}(0) \) has distance to \( H \) less than \( \epsilon \).
construct a smooth family of minimal surfaces immersed in $\mathbb{R}^3$. Following Perez and Ros [37], we show that the existence of this family is precluded by the maximum principle for minimal surfaces. Thus, the isometry must exist.

Francisco Martin, of the University of Granada, also kindly pointed out that our method can be adapted to prove a similar symmetry for screw-motion invariant genus-one helcoids.

3. Characterizing the Catenoid

In this section, I highlight recent work done in [4].

Throughout the literature, the catenoid has been characterized as the unique, non-flat minimal surface of revolution by Bonnet, unique minimal annulus of finite total curvature [38], [33], and unique, embedded index one minimal surface [17], [32]. Index one implies there exists a compactly supported variation that decreases area – that is, the catenoid is unstable. However, certain subdomains of the catenoid do not admit such variations and are stable. In our work, we consider the marginally stable portion of the catenoid – a surface that lies on the boundary between these two classes.

In particular, let $Cat_{MS} = \{x_1^2 + x_2^2 = \cosh^2 x_3\} \cap \{1 - x_3 \tanh x_3 > 0\}$ denote the marginally stable piece of the standard catenoid. Bernstein and I characterize $Cat_{MS}$ as the unique area minimizing annulus among smooth minimal annuli spanning the boundary of the slab $\Omega = \{1 - x_3 \tanh x_3 > 0\}$.

We rely on a classical result of Osserman and Schiffer that determined a convexity condition on the lengths of image circles of annuli [36]. Let $A(\Omega)$ denote the class of smooth minimal annuli spanning $\partial \Omega$. In this class, $Cat_{MS}$ is the unique area minimizer.

**Theorem 3.1.** Consider $\Sigma \in A(\Omega)$. Then

$$H^2(\Sigma) \geq H^2(C_{MS})$$

with equality if and only if $\Sigma$ is a translate of $C_{MS}$.

Here $H^2$ denotes the two dimensional Hausdorff measure.

For any $\Sigma \in A(\Omega)$, we may define $\text{Flux}(\Sigma) = \text{Flux}(\gamma)$ for $\gamma \subset \Sigma$ such that $|\gamma|$ is a generator of $H^1(\Sigma)$. This quantity is well defined (up to choice of orientation). An important property of the flux is that it sets a natural scale for a minimal annulus. In fact, up to rigid motions, each catenoid can be characterized entirely by its flux vector.

To that end, Bernstein and I show that given $\Sigma \in A(\Omega)$, there is an appropriate “matching catenoid” that provides an area lower bound for $\Sigma$. For this theorem, let $\Omega$ be any horizontal slab in $\mathbb{R}^3$ with $\partial \Omega = P_+ \cup P_-$ where $P_+ > P_-$.\n
**Theorem 3.2.** Fix $\Sigma \in A(\Omega)$. Let $P_0 = \{x_3 = h_0\} \subset \bar{\Omega}$ denote the plane that satisfies:

$$H^1(\Sigma \cap P_0) = \inf_{t \in (h_-, h_+)} H^1(\Sigma \cap \{x_3 = t\}).$$

Here $H^1(\Sigma \cap P_\pm)$ is defined as $\liminf_{t \to h_\pm} H^1(\Sigma \cap \{x_3 = t\})$ and likewise for $H^1(\Sigma \cap P_-)$. If $C$ is the vertical catenoid with $\text{Flux}(C) = (0, 0, F_3(\Sigma))$ and symmetric with respect to reflection through the plane $P_0$ then:

$$H^2(\Sigma) \geq H^2(C \cap \Omega)$$

with equality if and only if (a translate of) $\Sigma$ equals $C \cap \Omega$. Here $F_3(\Sigma)$ is the vertical component of $\text{Flux}(\Sigma)$.\n
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4. LOCAL PROPERTIES OF EMBEDDED MINIMAL DISKS

I outline the work done in [2, 5].

There are two fundamental models for embedded minimal disks with boundary on an extrinsic ball. Either the curvature is suitably small and on a subdisk the surface is graphical, or the curvature is large at an interior point and the disk has “helicoid-like” behavior. See [10, 11, 12, 13]. Bernstein and I give a condition for an embedded minimal disk to look like a piece of a helicoid. Namely, if such a disk has boundary in the boundary of a ball and has large curvature, then, in a smaller ball, it is bi-Lipschitz to a piece of a helicoid. Moreover, the Lipschitz constant can be chosen as close to 1 as desired (compare with Proposition 2 of [23]).

\textbf{Theorem 4.1.} Given \( \epsilon, R > 0 \) there exist \( s > 0 \) and \( R' \geq R \) so: Suppose \( 0 \in \Sigma' \) is a properly embedded minimal disk with \( \Sigma' \subset B_{R'}(0), \partial \Sigma' \subset \partial B_{R}(0) \) such that \( \sup_{B_{s}(0) \cap \Sigma'} |A|^2 \leq 4|A|^2(0) = Cs^{-2} \) for some constant \( C \). (Here \( A \) is the second fundamental form of \( \Sigma' \).) Then there exists \( \Omega \), a subset of some helicoid, so that \( \Sigma, \) the component of \( \Sigma' \cap B_R \) containing \( 0 \), is bi-Lipschitz with \( \Omega \) with the Lipschitz constant in \((1 - \epsilon, 1 + \epsilon)\).

While it is clear that minimal disks with large curvature must be close to the helicoid on some scale, the distortion from the helicoid on a larger scale can be quite extreme. For instance, in [34], Meeks and Weber “bend” the helicoid; they construct minimal disks where the axis is an arbitrary \( C^{1,1} \) curve.

Other examples of distortion are given in [9], [16], [20], [30], [31]. Each of these papers provides a sequence of minimal disks modeled on the helicoid, but where the the ratio between the scales (i.e. the measure of the tightness of the spiraling of the multi-graphs) at different points of the axis grows arbitrarily large. Importantly, such large distortions are, in some sense, global. Indeed, near points of large curvature a minimal disk looks like a piece of the helicoid with small distortion.

Using the example provided by Colding and Minicozzi in [9] and a compactness argument, Bernstein and I show that a result like Theorem 4.1 cannot hold on the outer scale \( R' \) or even on smaller scales.

\textbf{Theorem 4.2.} Given \( \epsilon > 0, 1 > \Omega > 0 \) and \( 1/2 > \gamma \geq 0 \) there exists an embedded minimal disk \( 0 \in \Sigma \) with \( \partial \Sigma \subset \partial B_{R'} \) and \( \sup_{B_{s}(0) \cap \Sigma'} |A|^2 \leq 4|A|^2(0) = Cs^{-2} \) for some constant \( C \) so: The component of \( B_{\Omega(R')^{1-\gamma}} \cap \Sigma \) containing \( 0 \) is not bi-Lipschitz to a piece of the helicoid with Lipschitz constant in \(((1 + \epsilon)^{-1}, 1 + \epsilon)\).

5. CURRENT AND FUTURE WORK

5.1. Existence of Higher Genus Helicoids. The existence of an embedded genus one helicoid arose naturally out of the study of a family of screw motion invariant genus one surfaces. Thus far, attempts to extend this method to the higher genus case have been unsuccessful. In fact, even the existence of a periodic example with higher genus in the quotient remains unknown.

Along with Nicos Kapouleas, from Brown University, I am attempting to establish the existence of higher genus helicoids. Based on our work so far, it seems highly unlikely that gluing methods alone will allow us to construct such examples. The gluing methodology, however, has provided some insight into this problem and we hope that by incorporating further construction techniques we will be able to succeed in constructing a higher genus helicoid.

While this problem is highly ambitious, if successful both the result and the methodology will be of immediate consequence in the community at large.

5.2. Construction of Embedded Constant Mean Curvature Surfaces. In [26] Kapouleas pioneered the method of gluing constructions for constant mean curvature (CMC) surfaces. This first paper used as its building blocks spheres and pieces of Delaunay surfaces, both of which are
rotationally symmetric CMC surfaces in $\mathbb{R}^3$. Kapouleas systematized and further developed his methodology when constructing more examples for CMC surfaces, this time by fusing Wente tori [27].

Although the new methodology has now been applied in many constructions, see [18], [29], [28], it has not yet been applied to the original CMC constructions. Kapouleas and I are presently expanding his earlier work to improve upon the estimates and simplify the setup. We anticipate these estimates will broadly expand the class of known embedded examples. In particular, in some cases unbounded ends of the surfaces can now be written as graphs over initial embedded surfaces. While previously these ends might bend an arbitrarily large amount, and thus destroy embeddedness, the better estimates can preserve embeddedness of the solution.

In addition to expanding on the number of embedded CMC surfaces in $\mathbb{R}^3$, we are also pursuing gluing constructions for CMC surfaces in higher dimensions. As in the first work of Kapouleas, we are considering gluing methods for rotationally symmetric building blocks – Delaunay hypersurfaces and hyperspheres – in $\mathbb{R}^n$.

5.3. **Properness of Embedded Minimal Surfaces.** In [14] Colding and Minicozzi provide a partial result to a conjecture of Calabi [7] that was more recently revisited by Yau [40]. In particular, they show that any complete, embedded minimal surface in $\mathbb{R}^3$ with finite topology is proper. Their proof relies on two fundamental ideas that are results of [10, 11, 12, 13]. First, the structure of an embedded minimal disk with boundary on the boundary of a ball is somewhat well understood. At the least, one can get appropriate chord-arc bounds on the interior of such surfaces. Second, large intrinsic embedded disks that have chord-arc bounds on very small scales will possess chord arc bounds on larger scales. This result follows by the so called “one-sided curvature estimate” [13], which essentially implies that minimal disks that are extrinsically close must be relatively flat.

With Bernstein and Giuseppe Tinaglia from Kings College, I am investigating the question of properness in the case of infinite topology planar domains. Given a complete, embedded planar domain, we hope to show the embedding is proper. Using estimates from Colding and Minicozzi [8], we are able to establish at least a weak understanding of embedded minimal planar domains with boundary in the boundary of a ball. The main difficulties in applying the techniques from [14] arise when trying to produce a similar one-sided curvature estimate for planar domains. The theorem from [13] cannot be improved for planar domains, as it has an easy counterexample in the catenoid. Our hope is that proving a weaker version of this one-sided curvature estimate is possible, and will allow us to determine weak chord-arc bounds for embedded planar domains.

**References**

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