Dartov's Theorem: describes how the various closed braid representatives of a same link isotopy type are related to each other, and as a consequence, gives an algorithm for two elements of $B_n$ to relate the same link.

Then (Dartov 1935)

Let $V, V'$ two closed braids which represent isotopic links, $V = V_0, V_1, \ldots, V_k = V'$ s.t. $V_i$ & $V_{i-1}$ differ by an operation which is either

- $E^+$, involving only positive edges

or

- $W$, involving only positive edges.

In the latter case, the link $V$ and $V'$ are isotopic. Indeed, delete any other part of the links.

In both cases, the link $V$ and $V'$ are isotopic. Indeed, delete any other part of the links.

Note: we can always pass from $V$ to $V'$ by means of $E^+$ but not necessarily preserving property of being a closed braid (i.e., all edges > 0).

Goal: given a sequence of moves $V \rightarrow V'$ (not high, closed braids), modify it so we have only positive edges at each stage, and do $E^+$ and $W$ moves only.

Recall, given a link, we can get rid of a neg edge by adding a sandwhich. ("$W"""")

Goal: do the same in 1-punch formula (along the moves $V \rightarrow V'$).

Case 1: if we pass from a link to another one by an $E^+$ operation, can deduce a way to relate the two of cases to consider depending on whether edge > 0, < 0.
**Lemma 1:** Can always assume all links appearing in the sequence of moves are not in general position (i.e., have no edge coplanar with \( E \)).

(Clear: move pts just a tiny bit)

**Lemma 2:**

\[
\begin{array}{c}
C \quad \begin{array}{c}
\text{can move a pt keeping edge } \alpha \\text{ by a sequence of } \varepsilon^+ \& \varepsilon^- \text{ moves.}
\end{array}
\end{array}
\]

\((\text{assume } a \& b \text{ are not in the rest of the link})\)

**Proof:** If \([a,\alpha] \) positive then obtain \((\varepsilon^+)^{-1} \) to move \( b \), then \( \alpha \).

If \([a,\alpha] \) negative:

\[
\begin{array}{c}
\text{1) } \varepsilon^+ \text{ to insert } a'
\end{array}
\]

\[
\begin{array}{c}
\text{2) } \varepsilon^- \text{ to insert } c \& d' \text{ (w/ } a, a' \text{)
\end{array}
\]

\[
\begin{array}{c}
\text{3) } \varepsilon^+ \text{ remove } a' \& b
\end{array}
\]

\[
\begin{array}{c}
\text{4) } (\varepsilon^+)^{-1} \text{ remove } d'.
\end{array}
\]

**Lemma 3:** Assume \( \mathcal{V} \) has a negative edge \([a,\alpha] \) on which sawnتحد can be carried in 2 different manners: by inserting \( \varepsilon^+ \) \& \( \varepsilon^- \) in order by a sequence of \( \varepsilon^+ \& \varepsilon^- \) moves. (Subdivision of \( E, a, \alpha \) all possible ways of having a given link into a closed link are equivalent up to \( \varepsilon \) moved)

**Proof:** can always refine a subdivision to a fine subdivision of \([a,\alpha]\), e.g., inserting an extra pts \( \varepsilon^+ (a, a') \) by operation \( \varepsilon^+ \& \varepsilon^- \).

If \( b \) is positive:

\[
\begin{array}{c}
\text{do } \varepsilon^+ \quad \text{do } \varepsilon^-
\end{array}
\]

\[
\begin{array}{c}
\text{pick } e \text{ in right arm, get close to } \Delta (a_i, q, b_i)
\end{array}
\]

\((\text{if } \alpha \text{ is positive, } \Delta (a_i, q, b_i) \text{ in half of the link})

\((\text{if } \beta \text{ is positive, } \Delta (a_i, q, b_i) \text{ in half of the link})

\text{Similarly,}

\[
\begin{array}{c}
\text{pick } e \text{ in right arm, get close to } \Delta (a_i, q, b_i)
\end{array}
\]

\((\text{if } b \text{ is positive, } \Delta (a_i, q, b_i) \text{ in half of the link})

\((\text{if } \alpha \text{ is positive, } \Delta (a_i, q, b_i) \text{ in half of the link})

\text{(if } \beta \text{ is positive, } \Delta (a_i, q, b_i) \text{ in half of the link})

\text{(if } \alpha \text{ is positive, } \Delta (a_i, q, b_i) \text{ in half of the link})
Then, left with: given an $E$-quaternion $V_1 \sim V_2$ the corresponding closed braids $\tilde{V}_1, \tilde{V}_2$ obtained by adding semiteeth on related by $\tilde{E}^+$ & $\tilde{W}$ moves. (i.e. can go from semitooth on edge $a_1$ to semitooth on neg of $a_1$)

so study all 8 case of $\tilde{E}_{ac}$ depending on sign of $ab, bc, ac$

By lemma 3, just need to be able to transform $ac$ into $ac$ on our favorite semitooth on ac (if $c \geq 0$) into same for $ab, bc$. (don't touch any semitooth we've put on other neg edges)

- 2 cases already taken care of:

\[
\begin{align*}
&\begin{array}{c}
\begin{array}{c}
\text{a} \\
\text{b}
\end{array}
\end{array}
\quad \begin{array}{c}
\begin{array}{c}
\text{c}
\end{array}
\end{array}
\quad \begin{array}{c}
\begin{array}{c}
\text{x}
\end{array}
\end{array}
\end{align*}
\]

\[
\begin{align*}
&\begin{array}{c}
\begin{array}{c}
\text{c}
\end{array}
\end{array}
\quad\begin{array}{c}
\begin{array}{c}
\text{a}
\end{array}
\end{array}
\end{align*}
\]

- e.g. if

\[
\begin{align*}
&\begin{array}{c}
\begin{array}{c}
\text{b}
\end{array}
\end{array}
\quad \begin{array}{c}
\begin{array}{c}
\text{x}
\end{array}
\end{array}
\quad \begin{array}{c}
\begin{array}{c}
\text{c}
\end{array}
\end{array}
\end{align*}
\]

- if $a$ & $b$ very close to each other, then $E$ common semitooth

ie. $p^k a_0 = a, a_1, ..., a_n = c$

$b_0 = b, b_1, ..., b_n = c$

$d_1, ..., d_n$ st

- $c_i$ $d_i, ..., d_n$ are added semitooth

- Each pyramid of vertices $a_i, a_i, b_i, b_i, d_i, d_i$ doesn't meet the considered links (including semitooth added or other edges) in any unraveled plaes

- $(aibi)$ not coplanar with $L$ (just chosen $aibi$ gene on same)
(just take a sawtooth on ac & "move" it over to bc).

Then: \( a, d, a_1 d_2 \ldots a_n d_n \) (broad cam to initial file)

\[ \text{if } a_{n+1} b_{n+1} > 0 : \quad \text{then } \begin{align*}
E_{a_{n+1}, d_n} & \quad \text{get } \quad \ldots d_{n+1}, a_{n+1}, b_{n+1}, d_n \ldots \\
E_{a_{n+1}, d_n} & \quad \text{else similarly } \quad E_{a_{n+1}, d_n}^{-1}
\end{align*} \]

get...\( \ldots d_{n+1}, a_{n+1}, b_{n+1}, d_n \ldots \)

The final files are small (valid because if our assembly or parts can’t reach the cutting file, then...)

since it assumes \( \text{bc} \) as b'

In general: subdivide \([a, b]\) into small subintervals of...

3 common sawtooths

Composition of \([a, b]\) = get them in infinitely many steps

Similarly in other cases:

\[ \text{e.g. } C \text{ } \quad \text{[call neg]} \]

if \( b \) close to \([a, c]\), find common sawtooth for \( ab \) & \( ab' \), and for \( bc \) & \( b'c \)

otherwise bring \( b \) to \( b' \) in small steps as above

0. \( C \) : \text{If a very close to } b \text{ then...}

Case where \( \text{b, c, } \) is negative

If not, subdivide \( ab \) into small steps & end up with sawtooth \( mab + bc \) by region of \( \text{W} \).
\[ a \rightarrow b \rightarrow c \quad \Rightarrow \quad a \rightarrow b \rightarrow c \]

is a saw halluc on \(a-b-c\)

staying very close to triangle \(abc\) (hence appendix \#1)

\(a^3\) twice \(h\) get

\[ \triangle abc \parallel ac \text{ very thin} \]

\(\Rightarrow\) when \(a^3\) to remove \(a, b, c\) from path

\[ a^3 \longrightarrow c \quad w \]

\[ a^3 \rightarrow c \quad \Rightarrow \quad a, b, c \]

\(\Rightarrow\) all cases end

---

**Conjecture:**

\(\hat{\beta}\) closed back congr. to \(\beta \in B_n\)

\(\beta' \quad \beta' \in B_n\)

\(\hat{\beta} \quad \hat{\beta'}\) name same vertex kth is by \(k\) th off

\(\beta = \beta_i \sim \beta_2 \sim \ldots \sim \beta_k = \beta'\)

\[ n_i = n \quad B_n \quad \longrightarrow \quad B_{n_i} = n' \]

st. each move is either (\#I):

\[ \beta_i \longrightarrow b \beta_i b^{-1}, \quad b \in B_n \]

\(n_i \quad \text{conj (trans)}\)

or (\#II):

\[ \beta_i \longrightarrow \beta_i \sigma_{n_i}^{\pm 1} \quad \text{stabilizer} \]

\(n_i \quad n_i + 1 \quad \text{or vice versa (duality)}\)

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Clearly \(n_\#_1, n_\#_2\) don't modify the link isotopy type – or shows picture.