Lecture 24 - May 15

Q: How many sympl. 4-folds admit Lefschetz fibration?
A: not so many, because need a slight generization: Lefschetz pencils.

E.g. \( X \subset \mathbb{CP}^n \) proj. surface, take generic linear proj. \( \mathbb{CP}^n \to \mathbb{CP}^d \)

\[ \{ x_0 = \alpha x_1 \} \subset \mathbb{CP}^d \]

"fibers" = intersection of \( X \) w/ pencil of hyperplanes

\( \{ x_0 = \alpha x_1 \} \subset \mathbb{CP}^1 \)

\( g \) generic fiber is a smooth proj. curve \( \subset X \)

some isolated fibers may be singular (can show: at most nodes, in generic situation).

This looks like the previous setup except \( f = \pi_{12} \) in condition

\[ \pi \times \mathbb{CP}^{n-2} \subset \mathbb{CP}^d \]

\( \pi \) has base \( B \), \( f \) is a fibration (finite set).

This is because all hyperplane \( \{ x_0 = \alpha x_1 \} \subset \mathbb{CP}^1 \) contain \( \{ x_0 = x_1 = 0 \} \),

so all fibers of \( f \) contain \( B \).

Def: \( X \to B = \{(z_1,z_2)\} \) finite set, \( f: X \to \mathbb{CP}^d \) is a Lefschetz pencil if

- near \( \mathbf{p} \in B \), \( f^{-1}(\mathbf{p}) \) is a union of \( \mathbb{CP}^1 \)
- outside \( B \), \( f \) is a L. fibration, i.e. isolated crit pts when \( z_1^2 + z_2^2 \).

Blowup construction

\[ \mathbb{CP}^2 : = \{(x,l) \in \mathbb{C}^2 \times \mathbb{CP}^1, x \in \mathbb{C}^2 \} \quad (\text{O(-1)}) \]

\[ \pi \downarrow \mathbb{CP}^2 \]

\( \pi \) is 1-1 except at \( 0 \), \( \pi^{-1}(0) = \mathbb{CP}^1 \)

Replace \( O \) by set of \( \mathbb{C}^1 \) line through it.

Then the pencil of lines through \( 0 \) in \( \mathbb{CP}^2 \) lift to a family of disjoint lines (fibers of \( \mathbb{CP}^2 \to \mathbb{CP}^1 \). The exceptional curve of the blowup \( E = \pi^{-1}(0) \) is the zero section of family bundle, intersects each fiber.

This is a geometric description. Can also "blow up topologically" (given curve, near a point, do this!), or synthetically (behavior: the sympl. form on \( \mathbb{CP}^2 \) depends on choice of a "size" param = synthetic area of the exceptional curve.)
By def. of a leftschek pencil, if we blow up $X$ at the base pts of $b_i$ (i.e., get $\hat{X} \xrightarrow{\pi} X$ a new 4-mdl), then $f$ extends over all of $\hat{X}$, giving a leftschek fibration $\hat{f} : \hat{X} \to S^2$, with distinguished sections $E_1, \ldots, E_n$ (the ex-curves of the blumps).

This is exactly the same as above, only on $\hat{S}^2$, near each $b_i$.

Note these exc. sections have normal bundle of deg $-1$.

Concretely, given a LF with section of square $-1$, can “blow down” to get a L. pencil.

Adaptation of the results about LFs:

- Monodromy: the distinguished sections $E_1, \ldots, E_n$ define $n$ marked pts on $\Sigma_g$ (the fiber of $f$). So monodromy now takes place in $\text{Map}_n(\Sigma_g)$ (be still consis of Dehn twists / vanishing cycle).

In fact, can do better: over $C = \mathbb{P}^1 - \{g+3\}$

- trivialize normal bundle to section $E_i \Rightarrow$ remove a small disk around each marked pt, get monodromy $\pi_1(C - \text{cir} f) \to \text{Map}(\Sigma_g, n)$ (the vanishing cycle now $\subset \Sigma_g, n$).

Can repeat monodromy along loops in $\text{aff} \text{ aff} \{C b_i\}$ different that $= \text{Id}$ near base pts.

- Monodromy $S$ on $= \text{ failure of global triviality = boundary twist}$

$$S = \prod_{i=1}^n S_i,$$

(Entwisted; we’ll see later why).

So, if genus $g \geq 2$ (or $g = 1, n = 1$) then

- gen $g$ LFs w/ $n$ distinguished $-1$-exch - isom.

- $\subset \Sigma_g, n$ (facts of $S$ as $\Sigma_1$ (Dehn twists) $\Sigma_g, n$. Pm, Hurwitz.
Consider the pencil of conics on $\mathbb{P}^2$ and the Plücker relation for Grassmannians.

- **Plane (Grassmann):** $\mathbb{G}(2,3)$.
- The $\mathbb{P}^2$ of conics is $3$-dimensional.
- The Plücker embedding $\mathbb{G}(2,3) \to \mathbb{P}^{15}$.

By definition, a conic is a curve of degree 2 in $\mathbb{P}^2$. The Grassmannian $\mathbb{G}(2,3)$ parameterizes linear subspaces of dimension 2 in $\mathbb{P}^3$.

### Example
- A conic in $\mathbb{P}^2$ can be given by the equation $Ax^2 + Bxy + Cy^2 + Dz^2 = 0$.
- The Plücker coordinates $e_1, e_2, e_3$ correspond to the intersection points of the linear subspace with three particular hyperplanes.

### Key Points
- The Grassmannian $\mathbb{G}(2,3)$ is a 15-dimensional projective space.
- The Plücker relations provide a way to represent conics in terms of linear subspaces.
- The pencil of conics through a given point is a line in $\mathbb{G}(2,3)$.

**Fix** $(\mathbb{P}^2, (x_0, y_0, z_0))$ & consider family of conics through $x_0$.
Relation w/ branched covers w/ single Hurwitz branched covers:

Consider the composition
\[ \phi = \pi \circ f : X \times \mathbb{P}^1 \rightarrow \mathbb{P}^1 \]

Fibers of \( \phi \equiv \) preimages by \( f \) of the fibers of \( \pi \) (which lie in \( \mathbb{P}^1 \)).

If we blow up \( \mathbb{P}^2 \) at \( (0:0:1) \), and blow up \( X \) at \( f^{-1}(0:0:1) \), we get
\[ \mathbb{P}^2 \xrightarrow{\mathcal{F}} F_+ \xrightarrow{\pi} \mathbb{P}^1 \]

Fact: \( \phi \) is a Lefschetz pencil.

In fact, singular fibers of \( \phi \equiv \) preimages of fibers of \( \pi \) that are tangent to \( D \) at a smooth pt of \( D \).

Note: \( d\phi = d(\pi \circ f) \) surjective unless \( df \) not surj. (ie. we"n along R)
and \( \text{im} \, df \cap \ker \, d\pi = \{0\} \)

\( \text{Tor} \) only when \( D \)

Tangent to fiber of \( \pi \).

Ex: near a cusp,
\[ \mathbb{C}^2 \xrightarrow{\phi} \{ y^2 = x^3 \} \]
\[ \begin{array}{c}
\pi \xrightarrow{f} \mathbb{C} \\
\times
\end{array} \]
brand, \( \text{Im} \, df \cap \ker \, d\pi \)

Computation: \( \phi : \mathbb{C}^2 \rightarrow \mathbb{C} \)

While, near a cusp,
\[ \mathbb{C}^2 \xrightarrow{\phi} \{ y^2 = x^3 \} \]
\[ \begin{array}{c}
\pi \xrightarrow{f} \mathbb{C} \\
\times
\end{array} \]

Double cover of \( \mathbb{C}^2 \) branched along \( D \) is \n\[ \mathbb{C} \]
\[ \{ z^2 = x - y^2 \} \]
\[ \mathbb{C} \]
Taking \n\[ x = y^2 + z^2 \]

I.e. local mod \( h^\phi \) is \n\[ \mathbb{C}^2 \xrightarrow{\phi} \mathbb{C} \]
\[ \begin{array}{c}
\pi \xrightarrow{f} \mathbb{C} \\
\times \end{array} \]

as expected for a L-pencil.

(\( \phi \) Donaldson's existence result for L-pencils can be deduced from the existence result for branched covers).
Lifting the monodromy: $E_1 \rightarrow \Sigma \rightarrow (\text{base of } B)$

Fix a base pt $q_0 \in \mathbb{C}P^1$, and $L = \pi^{-1}(q_0)$, the fibre over $q_0$.

The map $f_{L} : \Sigma \rightarrow \mathbb{C}P^1$ is a $n$-fold branched covering

with singularity at the pts of $D \cap \Sigma$ and monodromy

$\Theta_{L} : \pi_1(\Sigma - \{2n\text{nits}\}) \rightarrow \mathbb{Z}_{n}$.

Let $L = L - \pi^{-1}(L \cap E) = \text{disc containing the pt of } L \cap D$.

$\Sigma = \Sigma - V(E) = \text{complement of } 2n$ of base pt in fibre of $\phi$ (in $\mathbb{C}P^1$).

If we move to a different fibre of $\pi$, the intersection pts in $L \cap D$ move, and this changes the covering $f_{L} : \Sigma \rightarrow \mathbb{C}P^1$.

In particular, moving along a loop $g \in \pi_1(C - \text{int}(\pi_1 D))$, get the broad monodromy of $D$ along $g$ ($C' \subset B_D$), and the monodromy of $\phi$ along $g$ ($C' \subset \text{Nag}_{g,n}$).

Recall the lifting homeomorphism from a subgrp $\text{LBd} < B_D$ (liftable braids) to $\text{Nag}_{g,n}$ (depends on $\Theta_{L}$).

- The broad monodromy of $D$ takes values in the liftable subgrp $\text{LBd}$.
- The monodromies are related by.

\[ \pi_1(C - \text{int}(\pi_1 D)) \xrightarrow{\text{broad int of } D} \text{LBd} \xrightarrow{\text{lifting homem.}} \text{Nag}_{g,n} \]

In particular, half-twists (along arcs st. $2 < \text{branching in same sheet of covering}$)

- Dehn twists (along loop formed by 2 pts of the arc)
- while (half-twists)$^2$, (half-twists)$^3$ (w/ non-matching ends) $\rightarrow 1$.

This is actually often the best way to compute the monodromy of a L-pencil!!