We've seen: $X$ proj. surface, $f: X \to \mathbb{CP}^2$ generic projection.

- $D$ branch curve = simple Hurwitz curve w/ any nodes
  
  $\theta: \tau_1(\mathbb{CP}^2 - D) \to \mathbb{C}^n$ satisfying some conditions

- the data $(f^*\nu, \theta)$ is a purely algebraic description of the top of the covering...

Q: what if we start with any $(f^*\nu, \theta)$ satisfying the algebraic conditions

- i.e. D simple Hurwitz, $\theta: \tau_1(\mathbb{CP}^2 - D) \to \mathbb{C}^n$. What do we get?

A: a symplectic branched covering.

Symplectic geom: recall $(X^{2n}, \omega)$ is symplectic if $\omega \in \Omega^2(X)$

\[ d\omega = 0, \quad \omega^n > 0 \]

- Darboux's Thm: near any point, I local chart, in each

\[ \omega = dx_1 \wedge dy_1 + \ldots + dx_n \wedge dy_n. \]

- New's stability Thm: $(\omega_t)_{t \in \mathbb{R}}$ sympl. forms on $X$ compact nbd,

\[ (\omega_t) \in H^2(X, \mathbb{R}) \text{ compact} \]

\[ = \exists (\phi_t)_{t \in \mathbb{R}}: X \to \mathbb{R}^5. \quad \phi^* \omega_t = \omega_0. \]

Ex.: a Kahler form on a C nbd is symplectic

So, e.g., $\omega_{FS} = i \partial \bar{\partial} \log |z|^2$ on $\mathbb{CP}^n$ which $0 \leq \omega_{FS}$ alg.

gives a sympl. stc.

- The symplectic category is much richer. (Hat example: Tian-Yau 70's)

  E.g.: Hodge theory of a Kahler nbd has b, even

  But Gompf 1994: any finitely produced $G$ is $\pi_1(X^4, \omega)$.

So: not every symplectic nbd has a C stc; but is any stable

almost-C stc, i.e. $J \in \text{End}(TX), J^2 = -\text{Id}$,

\[ \omega(J, J) \] is a Kähler metric.

(In fact, any of any stable acs's is contractible).

The diffeo: no constraint on $DJ$ ("integrability")

\[ [T^{1,0}, T^{1,0}] \neq T^{1,0}; \quad \exists \beta \neq 0; \quad DJ \neq 0 \]

(Obstruction: Nijenhuis tensor)
Def: A 4-fold $(X^4, \omega_4)$ is symplectic. We say $\Phi: X \to Y$ is a symplectic branched covering if $\forall p \in X$, 3 local coordinates near $p$ are adapted (see below) in which $\Phi$ is one of the 3 models:

- $\mathbb{C}^3 \to \mathbb{C}^2$: $(x, y) \mapsto (x^3 - xy, y)$
- $\mathbb{C}^2 \to \mathbb{C}^2$: $(x, y) \mapsto (x^3, y)$
- $\mathbb{C}^2 \to \mathbb{C}^2$: $(x, y) \mapsto (x^3, y)$

Adapted coordinates:

- $\Phi: U \to \mathbb{C}^2$ local diffeo such that $(\phi_1, \phi_2)$ (the symplectic viewed in the coordinate) is positive on complex lines, i.e., $\phi_1 \phi_2(v, iv) > 0 \forall v \neq 0$.

Equivalently, any complex 2-form $\omega \in \Omega^2$ is a symplectic form if and only if it is an area form.

Rmk: So the branch cut $D \subseteq Y$ is a symplectic submanifold of $Y$.

Prop: If $\Phi: X \to Y$ symplectic branched covering then there exists a symplectic form $\omega$ on $X$, with $[\omega] = \Phi^*[\omega_Y]$; in fact, there is a canonical $\omega$ up to symplectic isometry.

Pf: $\Phi^*[\omega]$ is closed, nondegenerate outside of $R$, but degenerate (in direction of $\ker \Phi$) along $R$.

Claim: There exists a 2-form $\alpha$ such that $\alpha|_R > 0$ at every point of $R$.

Then, take $\omega = \Phi^*[\omega_Y] + \epsilon \alpha$ for $\epsilon > 0$ small enough.

$\omega$ is closed, and $\iota_X \omega = \Phi^* (\omega_Y + \epsilon \alpha) + 2 \epsilon \iota_X \alpha = 0$ everywhere.

0: Using $\epsilon$ small enough.

Properly $\epsilon \iota_X \alpha > 0$ on $R$.

Claim: $[\omega]$ is contact; note this is symplectically convex.

Thus, we can interpolate $\omega$ and any two $\omega$'s regardless of $[\omega]$ is compact.
Claim: calc. in local model. In both models, ker df = \langle x - axi \rangle.

Take \( \alpha = d(\chi_1(x) \chi_2(y)) u dv \)
\[
\int \int \ u = k_1 \times, \ v = k_2 \times \\
\text{cut-off functions for a "box"}
\]
so \( x_1 \equiv 1 \) along \( R \nabla \phi \chi_1(x) \chi_2(y) \)

& sum these over open cover of \( R \).

Next, definition. Any single Hurwitz curve \( D \subset \mathbb{P}^2 \) can be isotoped among simple Hurwitz curves to a symplectic subbundle.

NB. Complex curve an symplectic \( (TD = \langle v, iv \rangle) \omega | TD > 0 \omega | (v, iv) \rangle = \langle v, iv \rangle \)

but being sympl. is much easier at each pt, just want TD close to

\( \text{complex} \) than to anti-complex. Can denote by "up to go." at each pt... 

Proof. recall \( \mathbb{P}^2 - pt = \text{holb. span of } \mathcal{O}(1) \)

Rescale fiber directions:

\( \begin{pmatrix} x : y : z \end{pmatrix} \rightarrow \begin{pmatrix} x : y : \lambda z \end{pmatrix} \) give \( D_2 \) for \( \lambda \rightarrow 0 \), \( D_2 \) shrinks to a nbhd of the zero section and \( CV \) is \( C^1 \) outside of a nbhd of vertical tangents, i.e. \( TD_2 \) converges to 0-section as well.

Now: away from tangents, \( \omega | TD_2 > 0 \) because the zero section is a symplectic subbundle.

near tangents, ok by local model. 

In fact, this contr. is continued up to isotopy among sympl. subbundles:

if \( D, D' \) sympl. & Hurwitz iso to Hurwitz curves

\( \delta \) scale down the family to get an isotopy among sympl. H. curves.

Never, our branched covers w/ sympl. H. branch curves satisfy assumption of the prop.

Going:

To every \((\mathbb{F}^k, \theta)\) (satisfying the def. cond.) can assign a sympl. land \( (X, \omega) \) and a sympl. group \( f : X \rightarrow \mathbb{P}^2 \), these are canonically determined up to isotopy.
So... our missing branched covers correspond to sympl. 4-folds !!

Q: how many sympl. 4-folds can we get in this way?

Observe: (up to choice of normalization factor), standard \( W_{CP^2} \) has the
property that \( [u_{w,CP^2}] = \) generator of lattice \( H^2(CP^2, Z) = H^2(CP^2, R) \)
So for a sympl. covering of \( CP^2 \) we must have:
\[ [w] = [f^*u_{w,CP^2}] \] is always an integer Chow/Lefschetz class.

\[ \begin{align*}
\text{Thm.} \quad (X^4, w) \quad \text{compact symp. 4-dif, } [w] \text{ integer class} \\
\Rightarrow \exists \text{ all large enough integer } k, 3 \text{ sympl. branched covering } \text{ } f_k : X \rightarrow CP^2, \text{ with the following properties:}
\begin{itemize}
  \item the symp. form on } X \text{ induced by } f_k \& CP^2 \text{ is } \\
  \text{isotropic to } k \omega
  \item the branch curve } D_k \text{ is a simple } 4 \cdot \text{curves w/ cusps, }
  \text{nodes, and negative self } \text{cings}.
  \item for suff. large } k, 3 \text{ canonical ways of contracting } f_k, \text{ up to isom-op \& node cancellation}
\end{itemize}
\end{align*} \]

\[ \text{NB: unknown whether statement can be improved so that neg. nodes don't occur at all?} \]

\[ \text{w/o renormalization/cancellations must be performed compatibly w/ } \Theta \text{\ i.e. can only create a pair of nodes if } \frac{\Theta}{\Theta'} \text{ \& } \Theta(\partial, \partial') \text{ are disjoint transpositions. (else } X \text{ becomes singular)} \]

\[ \text{In fact,} \]
\[ \text{Thm (A. Kuklov-Shcherbak):} \]
\[ \text{If } D_1, D_2 \text{ single Hurwitz curves w/ } \pm \text{node} \& \text{ cusps, irreducible.} \]
\[ \text{If } \deg D_1 = \deg D_2, \text{ same #cusps, same (#+node - #neg nodes)} \]
\[ \text{then } D_1, D_2 \text{ are equ. up to } \{ - \text{isom-op of } H. \text{ curves} \}
\text{- creation/cancellation of nodes} \]
Let \( D = \text{set of pairs a family of } \Delta^2 \text{ into } \text{(halfknots)}^{1, \pm 2, 3} \). 

\( \text{a compatible } \theta : \pi_1(\mathbb{CP}^2 - D) \to \mathbb{C}^n \)

up to natural equivalence relations

(conjugacy, Hirzebruch equiv, node coincides 

\( J = \{ (x, c), \ c \text{ irreducible group} \} / \text{symplectomorphism} \)

Then we have natural maps \( D \to J \)

\( J \to \text{sequence for } k \geq 0 \) of 

\( \text{eff of } D \)

2 composition one way \( J \quad (x, c) \to (f \circ \theta, \theta) \to (x, fcw) \)

But the other way be anything a priori? Don't know 

If all symplectic \( k \geq 0 \) of \( \mathbb{CP}^2 \) are obtained by the method of \( \Gamma_k \) 

- Actually, believe not.


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Idea of them: given \( (x, c) \), choose \( j \) compatible acs

- \( L \) is line bundle, \( c \subset L \)

- \( \Delta \text{some mon } L \), w/ curvature -2i\pi c

- \( \Delta \text{operators on sections of } L \):

\( \Delta s = \frac{1}{2} (\Delta s + i \Delta s \cdot j) \)

If \( (x, c, j) \) kink in then \( L \) is an angle holon line bundle

\( \Rightarrow k \geq 0 \), \( \mathbb{CP}^2 \) has many holon sections (define an embedding \( x \to \mathbb{CP}^2 \))

So choose 3 sections generically give \( f : x \to (s_0(x), s_1(x), s_2(x)) \)

branched going \( \gamma \).

If \( j \) is only on a c.s. then \( \mathbb{CP}^2 \neq 0 \) and in fact \( \mathbb{CP}^2 \) holon sections of \( \mathbb{CP}^2 \)

(Blow holon for general!)

Still, Donald's observation: for \( k \geq 0 \), \( \mathbb{CP}^2 \) has "agree holon" sections, i.e. s.t.

\( \text{sup } |s| < \frac{c}{v_k} \text{ sup } |s| \); find 3 sections with good enough tranversely

propto so that \( (s_0, s_1, s_2) \) is a branched going in the desired direction.

In particular, want: \( - (s_0, s_1, s_2) \) dont vanish simultaneously

"agreement: 

\( - \Delta s = s_0 \Delta s_1, s_2 + \ldots \) vanish tranversely

\( \text{transversely neat} \)

- \( \Delta f \text{r vanish tranversely } \ldots \) & much more..."