X = \mathbb{C}^{n+1} \text{ complex proj. surface},
\text{Can assume } \mathbb{C}^{n+1} \cap X = \emptyset, \text{ so } p: \mathbb{C}^{n} \to \mathbb{C}^{n+1} \rightarrow \mathbb{C}^2 \text{ linear proj.}.
\text{P}_1X = f: X \to \mathbb{C}^2 \text{ well-def. map}, \text{ deg } f = \text{deg } X.

\text{We'll assume } X \text{ is in generic position w.r.t. } p. \text{ In fact this can be ensured by choosing } p \text{ well.}

\text{Prop: For a generic choice of the linear proj. } p, \text{ } f: X \to \mathbb{C}^2 \text{ is a branched (classical) covering whose branch curve has only nodes & analytic cup singularities.}

\text{In fact: (1) } R \subseteq X \text{ ramification curve is a smooth alg. curve in } X
\text{ $D = f(R) \subseteq \mathbb{C}^2$ discriminant curve
\text{ planar alg. curve given by}
\text{ $\{ z \in \mathbb{C}^2/ \# f^{-1}(z) < \text{ deg } X \}$}

\text{ local model: } \forall p \in \mathbb{C}^2, \exists \text{ local holomorphic on } \mathbb{C}^2 \ni p
\text{ in which } f \text{ is a generic pt of } R
\text{ "singularity"
\text{ $\text{deg } Df = 3x^2 - y = 0$ smooth
\text{ $D = f(R) = \{(-2z, 3z^2)\} = \{27z^2 = 4z^2\}$ analytic cup}

\text{ What do nodes come from?}
\text{ They correspond to 2 distinct pts of } R \text{ where single branching occurs and which happen to map to the same point in } \mathbb{C}^2.

\text{Status of the result is dubious. See Kollár & Kuchar 2000 for an attempt

\text{ Idea pf: Transversality theory — in fact the 3 local models are precisely those for generic holomorphic maps } \mathbb{C}^2 \to \mathbb{C}^2 \text{ in sing. theory, so could try to achieve them by perturbing } f.
\text{ However need to do that using finite dim. space of linear f.s. on } \mathbb{C}^N !!!}
If allow ourselves to reembed $X$ into a larger projective space (increasing the degree), I'm sure this works, otherwise... here's how the proof should go...

Get the proj to $\mathbb{P}^3$ by successively projecting $\mathbb{P}^{r+1}$ onto $\mathbb{P}^r$ ($\mathbb{P}^r \sim$ family of lines through chosen point $q$). $r = N - 1, \ldots, 2$

(at each stage, take $q \in X$ so proj into $\mathbb{P}^{r+1}$ is well-defined on $X$)

Actually, image of $X$ by previous projection...

...as long as $r < 5$, can assume proj is a diffeomorphism of $X$ with $\mathbb{P}^5$.

Indeed: 1. Binomial thm to $X^2 = 3$-dim family of lines (indexed by $\mathbb{P}^1 X$)

2. Line through 2 points of $X$ = dual $X \times X$.

The set of all $q$ on all such lines is of dimension $\leq 5 = d - r$.

a generic choice of $q \in \mathbb{P}^N$, ensure all lines through $q$ intersect $X$ in at most 3 pts., transversely.

For a generic proj $\mathbb{P}^5 \dashrightarrow \mathbb{P}^4$, image of $X$ will have double pts at most.

(in particular $3$-intertwined) because: can take $q \in X$ any line target to $X$

and $\{(x,0) / x \in L, 0 \text{ pt} \text{ im} \text{ 2 pts of } X \} \rightarrow \mathbb{P}^5$ is generically finite-to-one.

For generic $q$, $\exists$ finitely many $L$ through $q$ & $2 \text{ pts } X$. (Double pts of proj of $L$) & $3 \text{ pts on this hitting } X \text{ 3 times}$ is actually a locus of $4 \leq 3$ so generic $q$ give multiple pts.

Next, $\mathbb{P}^4 \dashrightarrow \mathbb{P}^3$: no longer an immersion, but a generic $q$ is a regular value of $\pi_4 : \mathbb{P}^4 \dashrightarrow \mathbb{P}^3$ .

$\{ (x,0) / x \in L, 0 \text{ pt } \text{ im } 2 \text{ pts of } X \} \rightarrow \mathbb{P}^4$ is generically finite-to-one.

Nest of the targets are single targets so control order 2 if $X$ is 2.

So we can avoid stationary targets, i.e. those with control order $\geq 3$.

Lemma: if the set of all stationary targets in $X$ is of dim $\leq 2$.

PF: at a "generic" pt of $X$ (where 2 nd order non-degenerate), then an 2 stationary targets if $X \subset \mathbb{P}^3$; none if $X \subset \mathbb{P}^4$, $2 \times 3$

set of pts of $X$ with infinitely many stationary targets.

$X$ osculates its target plane to order 3 at all dyadic $X$, hence of dim $\leq 1$ unless it's all of $X$ - but that happens only if $X = \text{linear \ } \mathbb{P}^2$ 

So can take $q \in \mathbb{P}^4$, i.e. finitely many single pts pass through $q$.

no stationary targets.

This controls the non-immersed points (also one can control the line through

search pts of $X$, to control self-intersections.)
Prop: For given \( y \in \mathbb{P}^3 \), image of \( X \) is \( Y \subset \mathbb{P}^3 \) with

- Self-intersections along a curve \( D \) ("double curve")
  \( (y \sim \{z_1, z_2 = 0\} \subset \mathbb{C}^3) \)

- Triple points
  \( \{z_1, z_2, z_3 = 0\} \)
  (three an immersed sing)

- "pinched" (or 'Whitney umbrella') \( \{z_1 = z_2 z_3\} \).
  (Locally proj from \( x \) is \( (x, y) \mapsto (xy, x^2, y) \))

So need to study how a surface \( Y \subset \mathbb{P}^3 \) projects to \( \mathbb{P}^2 \).

- First ignore the sing., i.e. assume \( Y \) embedded.

\( \{(x, y) \in \mathbb{P}^3 \times \mathbb{P}^2 : (x,y) \text{ on } Y \} \)

- \((x, y) \in \mathbb{P}^3 \times \mathbb{P}^2 \) smooth away from diagonal

\( \mathbb{P}^3 \) Take a regular value of this \( proj^2 \), \( q \notin Y \)

Then its preimage \( = \{y \in Y \text{ line though } q \text{ and } y \text{ target of } Y \} \)

= non-singular pts of \( proj^2 \) centered at \( q \)

is a smooth curve \( RC_y \).

The \( proj^2 \) of \( R \times \mathbb{P}^2 \equiv \text{line through } q \text{ which are tangent to } Y \).

Claim: a generic choice of \( q \) ensures that among the line through \( q \):

- Finitely many stationary targets, all of constant order = 3
  (not higher)

- Finitely many bitangents, all simple and 6th order

- No tritangents (line tangent to \( Y \) in 3 points).

Idea: dimension: above lemma = \( \{(x, l) \in \mathbb{P}^3 \times \mathbb{P}^2 : \text{ l stationary tangent}\} \)

is of dim \( \leq 3 \), so proj to 1st factor is generally finite-to-one

(\( \text{ i.e. finitely many stationary targets } \))

Similarly for bitangents

Katkova-Koltov claim some of these

\( \text{ are at most } \text{ dim } 2 \), i.e. don't hit any \( y \in \mathbb{P}^3 \).

The stationary targets of order 3 yield the cusps & points.

The simple bitangents yield the nodes,
Sing of $Y$: double curve, high $p$ not a problem if choose $q$ not on any target to the double curve (generically ok) nor in any of the target planes at high $p$.

Then $f: X \to \mathbb{CP}^2$ doesn't see these imm. comp. of $Y$ (e.g. $f$ unramified near high $p$).

- pinches: if $q \notin$ target come to the pinch (generically ok)
- then $f: \mathbb{CP}^2 \to \mathbb{CP}^2$ is loc. of degree 2 and has single branching.

Another viewpoint: $R = \{ x \in X \mid \text{Jac}_f(x) = 0 \}$, $df: T_X \to S_f^* \mathbb{CP}^2$

$\text{Jac}_f = \Lambda^2 df \in \mathcal{H}(\Lambda^2 T_X \otimes \Lambda^2 f^* \mathbb{CP}^2) = K_x \otimes O(3)|_X$

(think of $\text{Jac}_f \sim K_x \otimes dx_1 \otimes dx_2 \otimes ...$)

"canonical bundle" $\to 2$-form in $O(3)$

This line bundle is nontrivial & its section $\text{Jac}_f$ must vanish along $R$.

We can even it vanish transversely along a smooth curve $R$.

Get: $[R] = [K_x] + 3[H]$ by hyperplane section class.

Also, $cups = points$ when $f|_R$ not an involution $\equiv$ zero of $df|_R$.

This lets us compute various things:
- $\deg f = \deg X$
- $\deg D = [R], [H] = (K_x) \cdot [H] + 3[H] \cdot [H]$ 
- $2g(D) - 2 = (K_x + [R]) \cdot [R]$ (adjunction)
- $\# cups = 12 [H] \cdot [H] + 9 [K_x] \cdot [H] + 2 [K_x] \cdot [K_x] = e(X)$
- $\# nodes given by: \ g(D) = (d-1)(d-2)/2 - \# cups - \# nodes$

Now, characterize $f: X \to \mathbb{CP}^2$ by

1. branch curve $D \subset \mathbb{CP}^2$.
2. $\pi_1(\mathbb{CP}^2 - D)$ is $\mathbb{Z}_n$, $\deg f$.

Given 1, $\exists$ finitely many choice of 2 (if finitely generated, $\mathbb{Z}_n$ finite).