Def: \( C \subset \mathbb{CP}^2 \) closed oriented dim \( n \geq 2 \) sub-manif w/ isolated singularities

is a Hurwitz curve if

* \((0:0:1) \notin C\)
* \(C\) intersects transversely & positively the fibers of \( \pi: (x:y:z) \mapsto (x:y)\)
* except at finitely many pts \( p_1, \ldots, p_r \in C\)

(singularities & vertical tangencies)

Given any Hurwitz curve \( C \subset \mathbb{CP}^2 \), we can still define braid monodromy.

The "degree" of \( C \) is \( d = [C] \cdot [\text{line}] \geq 0 \) (intersection number \( d/\text{num}\) 2 pts \( \text{slice in } H_2(\mathbb{CP}^2, \mathbb{Z}) \approx \mathbb{Z} \)).

~ stabilization in \( \mathbb{B}_d \).

Usually one requires a bit more by prescribing a class of model behavior at \( p_i \).

Near each \( p_i \), find \( U_i \), a model curve \( \tilde{C} \subset \mathbb{C}^2 \) (in allowed class of models) and orientation-preserving local diffeo. s.t.

\[
\begin{align*}
\pi^{-1}(U_i) & \rightarrow \tilde{C}_i \\
\pi^{-1}(U_i) & \rightarrow \mathbb{C}^2
\end{align*}
\]

Important class of Hurwitz curves: Simple Hurwitz curve: such that

\[
\begin{align*}
\{ \text{ the preimages of the special pts are distinct} \\
\text{all vert. tangents are non-degenerate } & \text{; modelled on } y^2 = x^n \\
\text{all singular pts are modelled on } A_n-\text{sings } & \text{; modelled on } y^2 = x^{n+1} \times \prod_{n=1} \text{n-th node} \text{ (n>1)} \\
\} & \text{ n=2 ordinary comp}
\end{align*}
\]

Link: Our notes are algebraic, which is the most common setting.

Later we'll also allow "\( A_n\)-sings" modelled on \( y^2 = x^{n+1} \cdot n>1 \)

(non-algebraic: "micron image")

- a Hurwitz curve can always be perturbed so special pts lie in different fibers of \( \pi \); and an isometry of Hurwitz curve can be perturbed so this holds at all energies in the ideology. (so this extra requirement isn't much of an issue)
Prop (..., Kulikov-Kharlamov 2003):

For simple hyperplane curves, the braid monodromy \( \pi_1, (\mathbb{C}^2, x) \to \mathbb{B}_d \) determines the curve \( C \) uniquely up to isotopy of \( \mathbb{CP}^2 \) preserving the projection \( \pi \).

Expect this to hold in full generality (all \( H \)-curves); e.g. Kulikov-Kharlamov show this remains true if we allow \( A_n \) sing. modulo, or \( y^k = x^l : \forall k, l \geq 1 \).

Idea: The braid monodromy describes \( C \) above \( \mathbb{D}^2 \times (\mathbb{S}^1)^2 \) \( \sim \mathbb{S}^1 \times \mathbb{S}^1 \) up to fiber-preserving isotopy (faithfully).

Using constraints about what can happen near \( q_i \) to glue in some kind local model for rest of each special pt, to recover all of \( C \).

Graphy:

\[ \begin{array}{c}
\{ \text{simple Hurwitz curve in } \mathbb{CP}^2 \} \\
\downarrow 1-1 \\
\{ (b_1, \ldots, b_r) \in \mathbb{B}_d / \prod b_i = \Delta^2 \} \\
\downarrow \text{simult. conj.}
\end{array} \]

(\( \downarrow \) similarly if allow \( y^k = x^l \) or \( y^2 = x^n \) models).

Rank: Can extend this discussion to curves in \( \mathbb{C}^2, \mathbb{CP}^1 \times \mathbb{CP}^1 \), ... (with suitable modifications so braid monodromy makes sense).

Isotopy problem for Hurwitz curves:

- Every alg. curve is Hurwitz (positivity of intersection)
- Unless it's reduced or passes through pole of projection ...
- But simple Hurwitz curves which are not homotopic to any alg. curve.

\( x \) in \( \mathbb{CP}^2 \): various families of examples w/ noda & cusps \( (A_1, A_2) \)

Eg: curve of degree 18 w/ 31 cusps \( (A_2 \text{ singularities only}) \)

(Notthoon early 90s. Idea: branching along an annulus → get curve w/ only many different \( \pi_1(\mathbb{CP}^2, C) \); only finitely many alg.

Conj: A single Hurwitz curve in \( \mathbb{CP}^2 \) which is smooth or nodal \( (A_1 \text{ only}) \)

is isotopic through Hurwitz curves to an algebraic curve.

\( \leftrightarrow \) "symplectic isotopy problem"; proved by Siebel-7/2003 using Hol-curve methods, for smooth curve of deg ≤ 17)
Geom. approach uses J-hol. curve theory, but one could try by Gr theory: Conj: every point of $\Delta^2$ into halfblows or square of halfblows is Harnack conj. equivalent to unit of an alg. curve. (explicit list of model b.m.f.'s).

E.g., for smooth curve: \{smooth alg. curve\} $\subseteq \mathbb{P}(\text{homogeneous alg. polynomial})$ divisor (and same for those which are nondegenerate base of fibers of $\pi$), connected set so they all have the same b.m.f. up to Harnack & conj. Poonhavan has shown it is $\Delta^2 = (\sigma_1 \cdots \sigma_{d-1})^d$.

Isotopy conjecture says factor into halfblows.

(No, can't prove conj. by this method except $d = 2$, maybe $d = 3$).

- This phenomenon is specific to projective curves.

Thm (Kulikov-Khovanskii 2003):

- Given any $(b_1, \ldots, b_r)$ algebraic brds (i.e. monodromy of isolated speed points of alg. curve — e.g. positive powers of halfblows),

- Is an algebraic curve $C \subseteq \mathbb{P}^2$ s.t. the basic monodromy of $C \cap (\mathbb{P}^2 \times 0)$ can be realized by the factorization $b_1 \cdots b_r$.

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**Complex projective surfaces:**

$X \subseteq \mathbb{P}^N$ a alg. projective surface, smooth

- $X$ defined by alg. equation (e.g. cayley interesection)

- $X$ compact complex manifold, $L$ ample line bundle

- (i.e. $\sigma_L = [01]$) is Kahler form: $\omega = 2\pi \text{Im}(\omega)$ (\text{Riemannic})

- for $k \gg 0$, $L^k$ has sufficiently many holomorphic sections

- so that choosing a basis $s_0, \ldots, s_N \in H^0(L^k)$,

  $$X \rightarrow \mathbb{P}^N$$

  $$x \mapsto (s_0(x); \ldots; s_N(x))$$

  is an embedding & make $X \times \mathbb{P}^N$ proj.

  (kodaira embedding $\Im_K$)

- Now, consider a linear projection

  $$p: \mathbb{P}^N \times \mathbb{P}^{N-3} \rightarrow \mathbb{P}^2$$

  (in fact can choose any such

  $$[x_0; \ldots; x_N] \mapsto (x_0:x_1:x_2)$$

  proj. up to proj. linear transformations)

- Can assume $\mathbb{C}^{N-3} \times X = \emptyset$ (for dimensional reasons: $\dim X = 2 < \text{codim } \mathbb{C}^{N-3}$)

  (in fact: given $x \in X$, space of $\mathbb{C}^{N-3}$'s passing through $x$ has codim $3$ in $\mathbb{C}^{N-3}$ in $\mathbb{C}^{N-3}$)

  so all $\mathbb{C}^{N-3}$'s intersecting $X$ form a nice codim $1$ family, generic $\mathbb{C}^{N-3}$ meets $X$.
Then by induction, get a well-defined map $f: \mathbb{P}^2 \times X \to \mathbb{C}P^2$.

**NB:** Fibres of $p = \text{linear map } \mathbb{C}P^{n-2} \to \mathbb{C}P^n$ (passing through the given $\mathbb{C}P^{n-2}$) they intersect $X$ in $\{X\}. [\mathbb{C}P^{n-2}] = \text{deg}(X) \beta \ (\alpha \text{ class in } H_2(\mathbb{C}P^n, \mathbb{Z})) \ i.e. \ \text{deg}(f) = \text{deg}(X)$

- We'll assume $X$ is in generic position with $p$. In fact, this can be ensured by choosing $p$ well.

**Prop:** For a generic choice of the linear proj $p$, $f: X \to \mathbb{C}P^2$ is a branched covering whose branch curve has only node & cusp singularities.

**In fact:**

1. $R \subset X$ ramification curve is a smooth alg. curve $\subset X$ isomorphic to $\mathbb{C}P^1$.

   \[ D = f(R) \subset \mathbb{C}P^2 \] is the discriminant curve plane alg. curve $\cup$ cusp node.

2. Local model: $V(p \in X, E)$ local hom. onto $U(p) \subset X$ $U(p) = \mathbb{C}P^2$

   in which $f$ is $\begin{cases} (x, y) \leftrightarrow (x, y) & \text{if } p \not\in R \\ \text{locally diffeo.} \end{cases}$

   - $(x, y) \leftrightarrow (x^2, y)\quad$ as generic pts of $R$
   - "single branching"

   \[ (x, y) \mapsto (x^3 - xy, y) \]

   "cusp".

   \[ \text{here } R: \text{det } H_f = 3x^2 - y = 0 \text{ smooth } D = f(R) = \{(x^3, 3x^2) \} \{27z_0^2 - 4z_2^3\} \text{ cusp sing.} \]

   - Where do node come from?

     They correspond to 2 distinct pts of $R$ where single branching occurs and which happen to map to the same point in $\mathbb{C}P^2$.

Status of the result is dubious. See Kulkur & Kulkur 2000 for an attempt.