Lectr 17 - Wed April 19

→ Leftovers from lec. 16: Giroux's construction of open books

Braid monodromy of complex plane curves (Zariski, Moutheau, ...)

Setup:

\[ C \subset \mathbb{C}^2 \text{ complex algebraic plane curve (possibly singular!)} \]

\[ P(x, y) = 0. \]

Assume: \( \forall x \in C, P(x, \cdot) : P_x \in \mathbb{C}[y] \) is a nonzero polynomial of degree \( d \) (incl. \( x \)).

This means \( P(x, y) = y^d + Q_{d-1}(x) y^{d-1} + \ldots + Q_0(x) \)

for some \( Q_0, \ldots, Q_{d-1} \in \mathbb{C}[x] \).

Geometrically: \( C \) doesn't have any vertical asymptotic branches, i.e. the projective completion \( \overline{C} \subset \mathbb{P}^2 = \mathbb{P}^1 \times \mathbb{P}^1 \) does not pass through \((x, 0) \forall x \in C\).

Discriminant: \( \Delta(x) = \text{discr} \text{im} \text{it} \text{ of the degree } d \text{ polynomial } P_x \in \mathbb{C}[y] \)

(when coefficients depend polynomially on \( x \)!

\[ \Delta \in \mathbb{C}[x], \text{ its roots } \equiv \text{max} x \text{ s.t. } P_x \text{ has multiple roots in } y \]

\[ \equiv \text{value of } x \text{ s.t. } C \cap \{ y = 0 \} \text{ consists of fewer than } d \text{ distinct points} \]

Assume: \( C \) does not contain any multiple components, i.e. \( \Delta \not= 0 \).

(avoid: \( y^2 = 0 \) hodge line)

Let \( \{ q_1, \ldots, q_r \} \subset C \) := the distinct roots of \( \Delta \)

(\( r \) is often \( r < \text{deg } \Delta \)).

\[ \begin{array}{c}
\mathbb{C}^2 \\
\downarrow \pi \\
\mathbb{C} \\
\end{array} \]

The projection \( \pi : \mathbb{C}^2 \rightarrow \mathbb{C} \)

\( (x, y) \mapsto x \)

restricts to \( C \) as a (singular, ramified) \( d \)-fold covering, unramified over \( C \setminus \{q_1, \ldots, q_r\} \)

\( \{q_1, \ldots, q_r\} \) is \( r \) distinct points of \( C \) of degree \( d \), and by def\( ^* \), \( \{q_1, \ldots, q_r\} \) = \( \mathbb{P}^1 \) with fewer than \( d \) primes.
The only way \( C \cap \{ x < C \} \) can have fewer than \( d \) points is if

the adj. intersection multiplicity at one of these \( p_i \) (resp. intersection number) is \( > 1 \), which occurs iff \( C \) is either singular, or tangent to \( \nu \).

(mult. root of \( P \) \( \iff \) \( P = 0, \frac{\partial P}{\partial y} = 0 \); if \( \frac{\partial P}{\partial x} = 0 \), sing. pt, else tangency)

So, \( \{ q_i \} \) is projection of \( \{ \text{pts where } C \text{ is not smooth} \} \)

"special pts" of \( C = \{ \text{pts where } C \text{ is tangent to the fiber of } \nu \} \)." We have a natural map \( \sigma : C \setminus \{ q_1, \ldots, q_r \} \rightarrow \mathbb{C}d \) induced only for \( \nu \).

For \( x \in C \setminus \{ q_1, \ldots, q_r \}, \sigma(x) = \pi_{C}^{-1}(x) \in \mathbb{C}(\mathbb{R}) \) is an induced copy of \( d \) pts in the plane.

Fix a base point \( x_{a} \in C \setminus \{ q_1, \ldots, q_r \} \), and consider a loop \( \gamma \in \pi_{1}(C \setminus \{ q_1, \ldots, q_r \}, x_{a}) \rightarrow \rho(\gamma) = [\sigma_{\gamma}(\gamma)] \in \pi_{1}(\mathbb{C}d, \sigma(x_{a})) \)

\[ \mathbb{C}d \]

Defn: \( \rho : \pi_{1}(C \setminus \{ q_1, \ldots, q_r \}) \rightarrow \mathbb{C}d \) is the braid monodromy of \( C \)

(map on fundamental groups induced by \( \sigma \))

This depends on choice of an isom. \( \pi_{1}(\mathbb{C}d, \sigma(x_{a})) \rightarrow \mathbb{C}d \), induced by choice of homeo \( (\mathbb{C}, \pi_{1}^{-1}(x_{a})) \sim (\mathbb{R}^2, \{ 0 \}) \).

Diffeomorphism \( \sim \) relase \( P \) by its conjugation \( \text{by an inv. out. of } \mathbb{C}d \) (conjugation by some braid = "change of hand" of the fiber).

Exist: conic \( x^{2} + y^{2} = 1 \)

red path \( \frac{\sqrt{2}}{2} \)

y = 1 - x^{2} has a double root

\[ \text{iff } 1 - x^{2} = 0 \iff x = \pm 1 \]

- at \( x_{a} = 0 \), \( \sigma(0) = \{ \pm 1 \} \)

- consider the loop \( x(\theta) = (1 - e^{i\theta})^{1/2} \) (the square root will \( \Re x > 0 \)),

\[ 0 \leq \theta \leq 2\pi \]

\( \rightarrow \) above \( x(\theta) \) we have \( y^{2} = 1 - x(\theta)^{2} = e^{i\theta} \)

i.e. \( \sigma(x(\theta)) = \{ \pm e^{i\theta/2} \} \)

\( \Rightarrow \) The braid monodromy along this loop is \( \sigma_{1} \)

- similarly (b) symmetric around \( -1 \).

\[ \begin{array}{c}
\text{get:} \quad \sigma_{1} \end{array} \]
Non generally: if at some point the curve \( C \) is smoothly tangent to the fibre of \( \pi \) in a nondegenerate manner

\( \sigma \) monodromy around this \( q \) is a **half-twist** \( \in \text{BD} \)

Indeed, in nearby fibres of \( \pi \):

\[ y^2 = x, \text{ i.e. } y = \pm e^{i\theta / 2} \]

"local monodomy" is a half-twist.

= nothing happens

Monodomy along is the same braid up to

isomorphism \( \pi_1(C_0, \sigma(x)) \cong \pi_1(C_0, \sigma(x')) \).

induced by moving base pt along arc \( \sigma(x) \sim \sigma(x') \).

(so it's still a half-twist - it still exchanges strands CCW - but it may look more complicated than in the local picture because

\[ \sigma(x) \sim \text{trans} \sigma(x') \]

\[ \sigma(x) \sim \sigma(x') \]

**Ex. 2:** two lines \( y^2 = x^2 \)

monodromy around 0:

\( \sigma(x) = \{ \pm x \} \)

\( x = e^{i\theta} \leadsto \sigma(x) = \{ \pm e^{i\theta / 2} \} \) monodromy is \( \sigma^2 \)

Non generally, if at some point \( C \) has a node (transverse double pt)

both branches are transverse to the \( \text{proj} \pi \),

local monodromy = square of a half-twist

(\( \text{loc.} \))

Similarly, general types of singularities are recognizable from their braid monodromies! - braid monodromy is the natural way of describing the sing. of a plane curve & how they fit together.

Next: - setup for projective curves
- Zariski-Van Kampen