Def: $\Sigma$ oriented, $\gamma \subset \Sigma$ single closed curve

Dehn twist $T_\gamma \in \text{Hom}^+(\Sigma) = \{ \text{Id outside cylinder } U(x) \times [-\pi, \pi] \times S^1 | (x, \theta) \mapsto (x, \theta + \langle x, \eta \rangle) \text{ in cylinder} \}$

Notes:
- The isomorphism class of $\gamma$
- The deformation of $\gamma$
- The deformation of $\Sigma$ (but not of $\gamma$)
- Given $\circ \in \gamma$, $T_\gamma(\circ) = \text{near each intersection of } \alpha \cap \gamma$
- Assume $\alpha$ \perp $\gamma$ cut open $\alpha$ and insert a $\gamma'$ of $\gamma$, so that if we approach $\gamma$ by travelling along $\alpha$, we turn to our right when we hit $\gamma$

Thm. (Dehn 1938, Lickorish 1960s, ...)
Every oriented framing home of $\Sigma$ is isotopic to a product of Dehn twists.

$\text{Nap}^1(\Sigma)$, $\text{Nap}(\Sigma)$, and $\text{Nap}_n(\Sigma)$ are all generated by Dehn twists.

Strategy 1. Enough to prove it for $\text{Nap}^1(\Sigma)$. Indeed, recall kernel of $\text{Nap}(\Sigma) \rightarrow \text{Nap}^1(\Sigma)$ by Dehn twist/boundary curve

- Ker $\text{Nap}_n(\Sigma) \rightarrow \text{Nap}(\Sigma)$

Hence subgroup generated by $D$-twists in $\text{Nap}(\Sigma)$ maps onto $\text{Nap}(\Sigma)$ (by map) & contains kernel of $i_n$.

1. Given $h : \text{Hom}^+(\Sigma)$, consider disjoint curves $\alpha_1, \ldots, \alpha_n \subset \Sigma$, and $\gamma = h(\alpha_1)$. We'll find a product of Dehn twists $\phi_1, \ldots, \phi_n$ is isotopic to a curve disjoint from $\alpha_1, \ldots, \alpha_n$.

Then another product of $D$-twists $\phi_2$ s.t. $\phi_2(\gamma)$ is isotopic to $\alpha_1$.

Then $\phi_2 \phi_1 h$ (arising by a suitable isotopy) maps $\alpha_1$ to itself.

& we can assume reduced as Id: $\alpha_1 \rightarrow \alpha_1$. 

Then: cut $\Sigma_g$ open along $\alpha_1 \to \gamma_1$ to get $\Sigma_{g-1,2}$ 
and a homeo of it.

hence $[\phi_2][\phi_1][\phi] \in \text{Im}(\text{Nap}(\Sigma_{g-1,2}) \to \Sigma_g)$

and use induction on $g$ to conclude
(by above observation, reduce from $\Sigma_{g-1,2}$ to $\Sigma_{g-1}$
and for $\Sigma_0=S^2$ the statement is trivial).

In other words, what we’ll show is that the subgroup
$\text{gen}^2$ by Dehn twists act transitively on set of isotopy classes of
nonseparating simple closed curves.

Notation: $\tau = \text{subgrp of Homeo}^+ \text{ gen by Dehn twists & isotopies}$

$\gamma \leftrightarrow \gamma' \quad \text{if } \exists \phi \in \text{Homeo}^+, \phi(\gamma) = \gamma'$

$\gamma \leftrightarrow \gamma' \quad \text{if } \exists \phi \in \tau, \phi(\gamma) = \gamma'$

Lemmas: $\Sigma_{g_{1,2}}$ s.c.c. on $\Sigma_g$ with $|\gamma_1 \cap \gamma_2| = 1 \Rightarrow \gamma_1 \leftrightarrow \gamma_2$.

pf:

\[ \begin{array}{ccc}
\gamma_1 & \gamma_2 \\
\phi_1 & \phi_2 \\
\end{array} \]

observe $\gamma' \text{ is isotopic to } \tau_{\phi_1}(\gamma_1)$

but also to $\tau_{\phi_2^{-1}}(\gamma_2)$

so $\tau_{\phi_1} \tau_{\phi_2}(\gamma_1)$ is isotopic to $\gamma_2$.

Lemma 2: \( \alpha, \gamma \text{ single closed curves on } \Sigma_g, N=U(\alpha), \)

\( \exists \gamma' \in N, \gamma \leftrightarrow \gamma' \)

\( \gamma' \cap N = \emptyset \text{ or } |\gamma' \cap N| = 2 \) & then 2 strands have opposite orientations

pf: induction on $|\gamma \cap N|$.

- If $|\gamma \cap N| = 0$, \( \alpha, \gamma \text{ w/ opposite orientation, done. } \)
- If $|\gamma \cap N| = 1$, we Lemma 1, $\gamma \leftrightarrow \alpha$, take a pushoff of $\alpha$ disjoint from $\alpha$
- Otherwise we have one of 2 configurations; fixing orientation of $\alpha, \gamma$

Case 1:

\[ \begin{array}{ccc}
\alpha & \gamma \\
\beta & \gamma \\
\end{array} \]

$|\beta \cap \gamma| = 1 \Rightarrow \text{ by Lemma 1, } \gamma \leftrightarrow \beta$

$\alpha, \gamma \text{ w/ same orientation: }$

$|\alpha \cap \beta| < |\alpha \cap \gamma| \Rightarrow \text{ apply induction on } \beta$
3 consecutive intersections w/ alternating orientations can go either from $a \to c \to a$ or from $c \to a \to c$ along $\sigma$ without hitting $b$, assume it's $a \to c$.

Lemma 3: If $r$ simple closed curves $\alpha_1, ..., \alpha_r$ disjoint simple closed curves $\Rightarrow 3^r$ st. $r \leftrightarrow r'$, and $|r' \cap \alpha_i| = 0$, $a \neq 2$ w/ approch  $\forall i = 1, ..., r$.

If $r$, apply Lemma 2 repeatedly: first reduce intersections w/ $\alpha_1$ to 0 or 2.

Then reduce into w/ $\alpha_2$ to 0 or 2 — in the process, we either preserve the intersection w/ $\alpha_1$ (because we modify $\gamma$ only in a null of $\alpha_2$) or we remove them altogether (because we end up with $|r' \cap \alpha_2| = 1$ and replace $\gamma$ by a parallel copy of $\alpha_2$). Repeat process until $r = 0$.

We'll apply this lemma to $\alpha_1 \circlearrowleft \alpha_2 \circlearrowleft \alpha_3 \circlearrowleft$...

...and cut things into punctured tori $\Sigma_{i,1}$ along $\Sigma_i$'s.

Need a preliminary lemma about s.c.c.'s in $\Sigma_{i,1}$.

Lemma 4: If $\gamma$ simple closed curve in $\Sigma_{i,1}$ =

$\Rightarrow \gamma$ isotopic to a straight line

$\Rightarrow \gamma$ isotopic to a straight line through puncture

(by isotopy which is not $\text{Id}$ on $\partial_1 \Sigma_{i,1}$)

If $\gamma(a)$ — on $\Sigma_i = \mathbb{T}^2$ this is classical

— on $\Sigma_{i,1}$: first deform $\gamma$ to a straight line on $\mathbb{T}^2$;

more punctu in t-dependent way if needed so it's not hit.

Then translate all curves in isotopy so the punctu doesn't move.
(b) Similarly, clearly to a s.c.c. on \(\mathcal{T}^2\) by adding a disc 
whose core becomes a marked pt on \(\eta\).

Define scc to straight line (moving marked pt if needed).

Unde marker of marked pt by translations; remove small disc there.

\[\text{Lemma 5:} \quad \gamma = h(\alpha_i), \quad \eta \subset \kappa \ast (-\Sigma_g) \quad (\alpha_i, \Sigma_i \text{ as in pic. below})\]

\[\implies \exists \gamma' \text{ s.t. } \gamma' \subset \gamma', \quad |\gamma'| = 0 \quad \forall i, \]

and \(|\gamma' \ast \Sigma_i| = 0, \quad 2 \text{ w. opp. over } \Sigma_i.\)

\[\text{Pf:} \quad \text{Lemma 3 \implies } \exists \gamma' / |\gamma'| = 0 \text{ and } |\gamma' \ast \Sigma_i| = \text{ either } 0 \text{ or } 2 \text{ w. opp. over } \Sigma_i.\]

If \(|\gamma'| = 0\) we're done. Else \(\exists i / |\gamma'| = 2 \text{ w. opp. over } \Sigma_i.\)

1. \text{Assume } |\gamma' \ast \Sigma_i| = 0.

Then \(\gamma' \subset \Sigma_i\) bounded by \(\Sigma_i\)

Use \text{Lemma 4: isohyp } \gamma' \text{ to a straight line.}

\(|\gamma'| = 2 \text{ w. opp. over } \Sigma_i\).

\[\implies \text{alg. \# num. of (isohyp int) is } 2 > 0 \]

\[\implies \text{can ensure } |\gamma'| = 0 \text{ by isohyp.}\]

2. \text{Assume } |\gamma' \ast \Sigma_i| = 2:\n

The 2 ph of \(\gamma' \ast \Sigma_i\) split \(\gamma'\) into 2 arcs, \n
call m the part in \(\Sigma_i\) by \text{Lemma 4: isohyp } \gamma' \text{ so that } \gamma \text{ is a straight line. Again, alg. \# num. of } \gamma \text{ and } \alpha_i \text{ is } 0 \text{ or } 1 \]

\[\implies \text{can ensure } |\gamma'| = 0 \text{ by isohyp.}\]

\[\text{Lemma 6:} \quad |\gamma = h(\alpha_i) \text{ as above } = \gamma' \ast \alpha_i\]

\[\text{Pf:} \quad \text{start with } \gamma' \text{ given by } \text{Lemma 5: } \gamma' \subset \gamma', \quad |\gamma'| = 0, \quad |\gamma' \ast \Sigma_i| = 0 \text{ or } 2 \quad \text{marked.}\]

\[\implies \gamma' \subset \Sigma_g \ast \eta (\alpha_i \ast g_g \cdots \ast g_g) \equiv \Sigma_0, \Sigma_g \leftarrow 2g \quad \text{ boundaries } = \]

\[\text{every single closed curve on } \mathcal{T}^2 \text{ is spanning.} \quad 2 \text{ off of each } \Sigma_i.\]

\[\exists i \text{ s.t. the } 2 \text{ boundaries copy to } \alpha_i \text{ on } 2 \text{ sides of } \gamma' \]

(otherwise glue boundaries together to } \gamma' \text{ was spanning on } \mathcal{T}, \text{ contradicting since } \gamma' \text{ is one of } \gamma \text{ non-spanning})

\[\text{si } \alpha_i \text{ after opening along } \alpha_i, \beta_i \text{ join 2 sides of } \gamma', \text{ so } |\gamma' \ast \beta_i|\text{ is odd, } |\gamma' \ast \alpha_i| = 0, \quad |\gamma' \ast \Sigma_i| = 0 \text{ or } 2 \text{ w. i}.)\]
Lemma 4: an isopr $\gamma$ so $\gamma'$ or its path inside $\Sigma_{i,1}$ is a straight line.

$\mathbf{[} \mathbf{i} \mathbf{]} \mathbf{[} \mathbf{a} \mathbf{i} \mathbf{]} = 0 \Rightarrow$ the line is $\parallel \alpha_i$ and so intersects $\beta_i$ exactly once.

Hence $\gamma' \parallel \beta_i$

by Lemma 1.

- $\mathbf{[} \mathbf{i} \mathbf{]} \mathbf{[} \mathbf{a} \mathbf{i} \mathbf{]} = 1$: $\beta_i \cap \alpha_i = 1$ so $\beta_i \perp \alpha_i$

- if $\mathbf{[} \mathbf{i} \mathbf{]} \mathbf{[} \mathbf{a} \mathbf{i} \mathbf{]} = 1$ then $\beta_i \cap \alpha_i$ is a plane.

Consider if as explained at beginning: induction on $\mathbf{g}$, $\mathbf{g} = 0$ trivial. Assume $\mathbf{g} = 1$ ok.

- $\mathbf{L} \mathbf{6} \Rightarrow h(\alpha_1) \subset \alpha_1$, so $\exists$ par. $D.$ wrt $\phi$ st. $\phi(h(\alpha_1))$ is par. $h \alpha_1$.

Copy $\alpha_i$ with suitable isometry to assume $\phi h$ is Id on $\alpha_i$.

- $\alpha$ open along $\alpha_1 \Rightarrow \exists$ par. $h|_\Sigma_{E,2}$, and $\iota_\alpha: \Sigma_{E,2} \Rightarrow \Sigma_{E,1}$.

$i_\alpha(h) = \text{par. } D.$ wrt $h$ by induction hypothesis

$\Rightarrow \tilde{h} = (\iota \circ h \circ \iota_\alpha)^{-1}$.

$L_0 = \text{par. of } D.$ wrt $h$ & spin maps

$\iota_\alpha: \text{par. of } D.$ wrt $h$.

$\Rightarrow \tilde{h}$ is par. $D.$ wrt $h$, so $\phi h$ too, so $h$ too.