Lecture 10 - March 15 - Mapping class groups

Notation: $\Sigma_g = \text{closed orientable surface of genus } g$

$\Sigma_g, \Sigma_{g,r} = \text{orientable surface of genus } g \text{ w/ r boundary components}$

$z^0, \ldots, z^n = n \text{ fixed distinct pts on } \Sigma_g \text{ or } \Sigma_{g,r}$

$\text{Homeo}^+ (\Sigma_g) = \text{orientation-preserving homeos of } \Sigma_g \text{ (compact-open top.)}$

$\text{Homeo}^+ (\Sigma_{g,r}) = \text{ } \Sigma_{g,r} \text{ s.t. } \phi|_{\partial \Sigma_{g,r}} = \text{Id}$

$\text{Homeo}^+_n (\Sigma) = \{ \phi \in \text{Homeo}^+ (\Sigma) / \phi (z^0) = z^0 \text{ & } \forall i \}$

$\text{Homeo}^+_n (\Sigma) = \{ \phi \in \text{Homeo}^+ (\Sigma) / \phi (\{ z^0, \ldots, z^n \}) = \{ z^0, \ldots, z^n \} \}$

$\text{Nap} (\Sigma_g) = \text{Nap}(\Sigma), \text{ Nap} (\Sigma_{g,r}), \text{ Nap}_n (\Sigma_{g,r}), \text{ Nap}_{1,n} (\Sigma_{g,r})$

Mapping class groups $\leftrightarrow \text{Homeo}_n$ differ by $\text{Aut}_n$

Goal: understand the structure of these groups & how they relate to each other

Relating various flavors of mcgs:

Observe: $\Sigma \cong \Sigma' \Rightarrow (\text{a homeo of } \Sigma, \text{ which extends to } \Sigma')$

(by Id $\Rightarrow \Sigma' \cong \Sigma$). This induces both homeomorphisms $\text{Nap}(\Sigma) \cong \text{Nap}(\Sigma')$

- We'll see: $\text{Nap} (\Sigma_{g,r}) \cong \text{Nap} (\Sigma_g)$

1. Pinching vs. boundary components: $\text{Homeo}^+ (\Sigma_{g,r}) \cong \text{Homeo}^+_n (\Sigma_{g,r})$

This induces $\text{Homeo}^+ (\Sigma_{g,r}) \cong \text{Homeo}^+_n (\Sigma_{g,r})$

If $g \geq 2$, $r \geq 3$ if $g = 0$, $r > 3$, $\text{Nap}(\Sigma_{g,r}) \cong \text{Nap}_n (\Sigma_{g,r}) \cong \text{Nap}(\Sigma_g)$

(Warning?!) Central extension, kernel sent to $\mathbb{Z}$ with map $\phi(\text{Id})$ nontrivial and $\phi(\text{Id}) = \text{Id}_n$ if $g = 0$, $r = 2$

For $g = 0$, $r = 2$, $\text{Nap}(\Sigma_{0,2}) \cong \mathbb{Z}$ (both twist maps on $\Sigma$)

$g = 0$, $r = 1$, $\text{Nap}(\Sigma_{0,1}) = 1$ (seen: $\text{Nap}_{1,1} (\Sigma_{0,1}) = B_n$)

Proof (idea): given a homeo of $\Sigma_g$, $\phi(z^0) = z^0$, can approximate it by a homeo which is Id on a small old $D(\eta)$

Idea: for small $\varepsilon > 0$

build $q_{\varepsilon}$ as follows:

$D(\varepsilon) = C_\varepsilon \text{ for } \phi$:

$D_\varepsilon = C_\varepsilon \text{ radius } \varepsilon$

School idea was (given 2 simple closed curves $C, C'$ in the plane, any homeo $h: C \leftrightarrow C'$ extends to a homeo of the interior region)

Note that $D_{\eta} \subset \phi(D_{\varepsilon})$
get a homeo $\overline{D_\varepsilon} \to \overline{\phi(D_\varepsilon)}$
which agrees with $\phi|_{\partial D_\varepsilon} : \partial D_\varepsilon \to \partial \phi(D_\varepsilon)$ on boundary
is $\text{Id}$ over $D_\eta$

Then replace $\phi$ by this homeo inside $D_\varepsilon$
get $\phi_\varepsilon$; and in compact-open topology $\phi_\varepsilon \to \phi$

$= \phi, \phi_\varepsilon$ in same con. component of $\text{Hom}_+^r(\Sigma_g)$ if $\varepsilon$ small enough!

Now, maybe the disc of radius $\eta$ doesn't contain the disc $B_i$ but:
conjugate $\phi_\varepsilon$ with family $(r, \theta) \mapsto (f_\varepsilon(r), \theta)$
in $\Sigma_g \times \Sigma_g$

$D(r) \supset D(\eta) = B_i$

so it's $\text{Id}$ on a larger ad larger disc (go from $\varepsilon = 0$ to $\varepsilon$ sf.
when $\varepsilon < \eta$

then conjugately homeo takes $B_i$ into $D(\eta)$

This implies surjectivity of $\varepsilon$: $\text{Nap}(\Sigma_g) \to \text{Nap}_r(\Sigma_g)$

we found a homeo fixing $B_i$ pointwise in the same
con. component of $\text{Hom}_+^r(\Sigma_g)$ as $\phi$.

Now understand $\ker(\varepsilon)$: assume $\phi_0, \phi_1 = \text{Id}$ on $B_i$, in same con. component of $\text{Nap}_r(\Sigma_g)$ ie joined by $(\phi_\varepsilon)_{\varepsilon \in [0,1]}$ arc in $\text{Hom}_+^r(\Sigma_g)$.

Idea: can similarly approximate $(\phi_\varepsilon)$ by an arc $\varepsilon$:$\phi_\varepsilon$ maps a small disc $D(\eta)$ to itself -- but can't assume $\phi_\varepsilon|_{\partial D(\eta)} = \text{Id}$ $\forall \varepsilon$.

Hence, can assume $\phi_\varepsilon|_{\partial D(\eta)}$ is a rotation. (and $\phi_0 = \phi_1 \equiv \text{Id}$ on $B_i$)

Proceed as above, but look at the homeo $\Phi : (t, z) \mapsto (t, \phi_\varepsilon(z))$
on $[0,1] \times \overline{D_\varepsilon}$, $D_\varepsilon \subset B_i$; and we $\phi_0 = \phi_1 = \text{Id}$ on $B_i$ to think of it as a homeo from $S^1 \times \overline{D_\varepsilon}$ to $\overline{\Phi(S^1 \times \overline{D_\varepsilon})}$.

For $\eta < \varepsilon$, $S^1 \times (D_\varepsilon - D_\eta)$ and $\overline{\Phi(S^1 \times \overline{D_\varepsilon})} = S^1 \times \overline{D_\eta}$ at again homeomorphic
by a homeo $(t, z) \mapsto (t, \psi_\varepsilon(z))$

Claim: we can ensure

\[
\begin{align*}
\psi_{t=0} &= \text{Id} \\
\psi_t|_{\partial D_\varepsilon} &= \phi_t|_{\partial D_\varepsilon} \\
\psi_t|_{\partial D_\eta} &= \text{rotation}
\end{align*}
\]

- first conjugate each $\psi_t$ with $\psi_0 : D_\varepsilon - D_\eta \to D_\varepsilon - D_\eta$, to ensure (1)
- then ever (2) by equating \( \psi_t \) with a homeo \((t, r, \theta) \mapsto (t, r, f_t(\theta)) \)
  when \( f_t : S^1 \to S^1 \) is the diffeomorphism \( t \mapsto \psi_t \) and \( \psi_t \) on \( C \).

- then ever (3) by looking at \( \psi_t \) on \( S^1 \times C \to S^1 \times C \)
  \((t, \eta, \theta) \mapsto (t, \eta, \theta + S(t, \theta)) \)

- \( S : S^1 \times C \to \R \) continuous,
- \( \theta + S(t, \theta) \) strictly a function of \( \theta \)
- \( S \) lifts to a function on \( \R \) over \( \R^+ \times \R^+ \) periodic on \( C \)
  \( \text{however } S(t+1, \theta) = S(t, \theta) + k \text{ for some } k \in \Z ! \)
- WLOG \( S(t=0, \theta) = 0, S(t=1, \theta) = k \text{ since } \psi_t = \text{Id}. \)

Then we can interpet the \( \Theta \) as \( \Theta \circ S(t, \theta) \) and \( \Theta \circ \theta + kt \) via
\( \psi_t \) the same properly (take: \( \lambda S(t, \theta) + (1-\lambda)kt \)).

- \( \Theta \) om a with \( \frac{\partial}{\partial t} \times (t, \theta, \varphi + \chi(t, \theta - t)) \)

- \( f_t \) as for \( t = 0 \) \( \varphi = 0 \) \( \Psi_t = \text{Id} \)
- \( r = 1, \) \( \lambda = 1 \)
- \( r = 0, \)
- \( r = \eta, \) \( (t, \eta, \theta + S(t, \theta) - kt) \)

\( \Psi_t \)
- replace \( \psi_t \) by \( \psi_t \) by \( \psi_t \).

Now, glue together \( \phi_t \) outside \( D_\psi \)
- \( \psi_t \) on \( D_\psi \times D_\eta \)
- rotation by \( kt \) on \( D_\eta \).

\( \phi_t \) in compact-open topology

As before, conjugating by radial scaling we can assume \( \phi_t, \psi_t = \text{Id} \) on a
- fixed given disc (e.g. \( B^3 \)) instead of \( D_\eta \).

- Now assume \( \phi \in \ker(\phi) \) and take an arc in \( \ker(\phi) \) from \( \phi_0 = \phi \) to \( \phi_1 = \text{Id} \)

- we above argued to define it to an arc \( \phi \mapsto \text{Id} \) such that \( \phi_t \) is a
glotten near \( \phi \).

Replace \( \circ \) \( \phi_t \) by \( \circ \), for all \( t \), near each \( \phi \)

\( \Rightarrow \) get an arc \( \phi \to \text{T} \) when maps inside \( \text{Home}^+(\varphi, r) \)

\( \text{Hence the } \text{T} \text{ twin maps generate } \ker(\phi) \)

\( \text{NB: the clearly commute, individual in } N(\varphi, \varphi, \varphi) \) - can enlarge \( \varphi \) - isometry so

\[ \sup(\text{twist}) \in \text{WS} \]
So \( \ker (\iota) = \text{Im}(\iota) \), \( \text{Im}(\iota) \to \text{Nap}(\Sigma_r) \), \( \text{Nap}(\Sigma_r) \), \( \text{claim if } g \geq 1 \text{ or } g = 0, r \geq 3 \), \( \text{this map is injective.} \)

To see this look at induced action of \( \text{Nap}(\Sigma_r) \) on \( \pi_1(\Sigma_r, p_i) \)

\( \text{This is a free gp on } 2g + r - 1 \text{ generators, hence nonabelian if } g \geq 1 \text{ or } g = 0, r \geq 3 \)

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and twisted map at \( \partial M \) by conjugation by \( i \), hence nonabelian

\( \text{To get } \mathbb{Z}^r \to \text{Nap}(\Sigma_r). \)

A more careful way to think about it: the above argument \((\psi_0, \psi) \to (\psi_0, \epsilon)\)

might fail when \( \phi_0, \phi_1 \Gamma \Sigma_i \) are abelian (not nec. \( \text{Id} \))

shows:

\[ \text{Def: } \text{Nap}_r(\Sigma) = \pi_0\{ \phi \in \text{Homeo}^+_\text{m}, \phi(z_i) = z_i \} = \pi_0\{ \phi \in \text{Homeo}^+_{\text{m}} \}
\]

\( \phi = \text{relamination in } \)

\( \text{eg. } \Phi_i \text{ a nbhd of } z_i \}

Then \( \exists \text{ bc. hdl relamination } \{ \phi \mid \phi = \text{Id on } B_i \} \subset \{ \phi \mid \phi = \text{relamination on } \Sigma \}
\]

\[ \text{For hdl to } \Phi_i \]

\( \text{and } \}

\[ \pi_1(\Sigma, B_i) \to \pi_1(\Sigma_i) \to \pi_0\{ \phi \mid \phi = \text{Id on } B_i \} \to \pi_0\{ \phi \mid \phi = \text{relation on } B_i \} \to 1
\]

\( \mathbb{Z}^r \to \text{Nap}(\Sigma_r) \)

\( \text{by def}\)

\( \text{one can show this map is zero if } g \geq 1 \text{ or } g = 0, r \geq 3 \)

\( \text{by looking at action on boundary}\)

\( \text{real boundary}\)

\[ \text{Forgetting marked pts. } \text{Nap}_r(\Sigma_r) \to \text{Nap}(\Sigma_r) \text{ induced by inclusion}\]

\[ \text{The evaluation map } \psi_0 \to (\psi_0(z_i)) \text{ defines bc. hdl relamns}\]

\[ \text{Hom}_n(\Sigma) \to \text{Hom}_n(\Sigma) \]

\[ \text{Hom}_n(\Sigma) \to \text{Hom}_n(\Sigma) \]

\( \text{in can } \Sigma = D^2, \text{ this is exactly what we used to prove } \mathbb{P}_n = \text{Map}_n(D^2)_1 \)

The argument is the same here!!

\[ \text{for } \mathbb{P}_n = \text{Map}_n(D^2)_1 \text{ homs. } \]
This induces a L.E.S.:
\[ \cdots \rightarrow \pi_1 \operatorname{Homeo}^+(\Sigma) \rightarrow \pi_1 \tilde{\text{Nap}}(\Sigma) \rightarrow \pi_0 \operatorname{Homeo}^+(\Sigma) \xrightarrow{\mathcal{S}} \pi_0 \tilde{\text{Nap}}(\Sigma) \rightarrow \pi_0 \tilde{\text{P}}_n(\Sigma) \rightarrow \cdots \]

where \( \tilde{\text{P}}_n(\Sigma) \) is the group of \( n \)-th order pure braids of \( \Sigma \).

The induced map \( \mathcal{S} \) sends a pure braid \( \beta = \beta_t \cdot \beta_0 \) to the \( n \)-th iterated suspension of \( \beta_0 \), where \( \beta_t \) is a pure braid fixing a subset of the punctures.

Then the map \( i_* : \text{Nap}_n(\Sigma, g, r) \rightarrow \text{Nap}(\Sigma, g, r) \) is a natural isomorphism, with \( \ker i_* = \text{Im}(\mathcal{S}) \) if \( g \geq 2 \) or \( r = 1 \), and

\[ \left\{ \begin{array}{ll}
\pi_n(\Sigma) & \text{if } \Sigma = T^2, \ n \geq 2 \\
\pi_n(\Sigma)/\text{center} & \text{if } \Sigma = S^2, \ n \geq 3
\end{array} \right. \]

The same statement holds for \( \cdots \rightarrow B_n(\Sigma) \rightarrow \text{Nap}_n(\Sigma) \xrightarrow{\mathcal{S}} \text{Nap}(\Sigma) \rightarrow \text{P}_n(\Sigma) \rightarrow \cdots \)

with \( \ker i_* = B_n(\Sigma) \), and \( B_n(\Sigma)/\text{center} \).

Note of the above follows from the L.E.S. induced by ev. fibration; we just need to understand \( \text{Im}(\mathcal{S}) = \text{P}_n(\Sigma)/\ker i_* \).