Geometric Ramsey Theory

Andrew Suk
MIT

January 14, 2013
A $k$-uniform hypergraph $H = (V, E)$, $V$ is the vertex set, and edge set $E \subset \binom{V}{k}$. We define a **clique** in $H$ to be a sub-hypergraph for which all $k$-tuples belong to $E$, and an **independent set** in $H$ to be a sub-hypergraph for which all $k$-tuples not in $E$. 
A $k$-uniform hypergraph $H = (V, E)$, $V$ is the vertex set, and edge set $E \subseteq \binom{V}{k}$. We define a **clique** in $H$ to be a sub-hypergraph for which all $k$-tuples belong to $E$, and an **independent set** in $H$ to be a sub-hypergraph for which all $k$-tuples not in $E$. 
A $k$-uniform hypergraph $H = (V, E)$, $V$ is the vertex set, and edge set $E \subseteq \binom{V}{k}$. We define a clique in $H$ to be a sub-hypergraph for which all $k$-tuples belong to $E$, and an independent set in $H$ to be a sub-hypergraph for which all $k$-tuples not in $E$.
A $k$-uniform hypergraph $H = (V, E)$, $V$ is the vertex set, and edge set $E \subset \binom{V}{k}$. We define a **clique** in $H$ to be a sub-hypergraph for which all $k$-tuples belong to $E$, and an **independent set** in $H$ to be a sub-hypergraph for which all $k$-tuples not in $E$. 
A $k$-uniform hypergraph $H = (V, E)$, $V$ is the vertex set, and edge set $E \subset \binom{V}{k}$. We define a **clique** in $H$ to be a sub-hypergraph for which all $k$-tuples belong to $E$, and an **independent set** in $H$ to be a sub-hypergraph for which all $k$-tuples not in $E$. 
A $k$-uniform hypergraph $H = (V, E)$, $V$ is the vertex set, and edge set $E \subset \binom{V}{k}$. We define a clique in $H$ to be a sub-hypergraph for which all $k$-tuples belong to $E$, and an independent set in $H$ to be a sub-hypergraph for which all $k$-tuples not in $E$. 
A $k$-uniform hypergraph $H = (V, E)$, $V$ is the vertex set, and edge set $E \subset \binom{V}{k}$. We define a **clique** in $H$ to be a sub-hypergraph for which all $k$-tuples belong to $E$, and an **independent set** in $H$ to be a sub-hypergraph for which all $k$-tuples not in $E$. 
A $k$-uniform hypergraph $H = (V, E)$, $V$ is the vertex set, and edge set $E \subseteq \binom{V}{k}$. We define a clique in $H$ to be a sub-hypergraph for which all $k$-tuples belong to $E$, and an independent set in $H$ to be a sub-hypergraph for which all $k$-tuples not in $E$. 
A $k$-uniform hypergraph $H = (V, E)$, $V$ is the vertex set, and edge set $E \subset \binom{V}{k}$. We define a clique in $H$ to be a sub-hypergraph for which all $k$-tuples belong to $E$, and an independent set in $H$ to be a sub-hypergraph for which all $k$-tuples not in $E$. 
A $k$-uniform hypergraph $H = (V, E)$, $V$ is the vertex set, and edge set $E \subset \binom{V}{k}$. We define a clique in $H$ to be a sub-hypergraph for which all $k$-tuples belong to $E$, and an independent set in $H$ to be a sub-hypergraph for which all $k$-tuples not in $E$. 
A $k$-uniform hypergraph $H = (V, E)$, $V$ is the vertex set, and edge set $E \subset \binom{V}{k}$. We define a **clique** in $H$ to be a sub-hypergraph for which all $k$-tuples belong to $E$, and an **independent set** in $H$ to be a sub-hypergraph for which all $k$-tuples not in $E$. 
Definition

We define the Ramsey number $R_k(n)$ to be the minimum integer $N$ such that any $N$-vertex $k$-uniform hypergraph $H$ contains either a clique or an independent set of size $n$.

Theorem (Ramsey 1930)

For all $k, n$, the Ramsey number $R_k(n)$ is finite.

Estimate $R_k(n)$, $k$ fixed and $n \to \infty$. 
Known estimates

**Theorem (Erdős-Szekeres 1935, Erdős 1947)**

\[ 2^{n/2} \leq R_2(n) \leq 2^{2n} . \]

**Theorem (Erdős-Rado 1952, Erdős-Hajnal 1960's)**

\[ 2^{cn^2} \leq R_3(n) \leq 2^{2c'n} . \]

\[ t_{k-1}(cn^2) \leq R_k(n) \leq t_k(c'n) . \]

*Tower function* \( t_i(x) \) is given by \( t_1(x) = x \) and \( t_{i+1}(x) = 2^{t_i(x)} \).
Problem (Esther Klein 1930’s)

Given an integer $n$, does there exist a number $ES(n)$, such that any set of at least $ES(n)$ points in the plane in general position, contains $n$ members in convex position?
Problem (Esther Klein 1930's)

*Given an integer $n$, does there exist a number $ES(n)$, such that any set of at least $ES(n)$ points in the plane in general position, contains $n$ members in convex position?*
ES(n) exists

\[ V = \{N \text{ points in the plane}\}, \]
\[ E = \{\text{triples having a clockwise orientation}\}. \]

**Observation:** Any subset of points for which every triple has the same orientation, must be in convex position.

\[ ES(n) \leq R_3(n) \leq 2^{2c' n}. \]

Can we do better?
$ES(n)$ exists

$V = \{N \text{ points in the plane}\}$,
$E = \{\text{triples having a clockwise orientation}\}$.

**Observation:** Any subset of points for which every triple has the same orientation, must be in convex position.

$$ES(n) \leq R_3(n) \leq 2^{2^c n}.$$ 

Can we do better?
Observation: Any subset of points for which every triple has the same orientation, must be in convex position.

\[ ES(n) \leq R_3(n) \leq 2^{2c'n} \]

Can we do better?
Observation: Any subset of points for which every triple has the same orientation, must be in convex position.

\[ V = \{ \text{N points in the plane} \}, \]
\[ E = \{ \text{triples having a clockwise orientation} \}. \]

\[ ES(n) \leq R_3(n) \leq 2^{2^{c'n}}. \]

Can we do better?
Observation: Any subset of points for which every triple has the same orientation, must be in convex position.

\[ V = \{N \text{ points in the plane}\}, \]
\[ E = \{\text{triples having a clockwise orientation}\}. \]

\[ ES(n) \leq R_3(n) \leq 2^{2c'n}. \]

Can we do better?
Theorem (Erdős-Szekeres 1935)

\[ 2^{n-2} + 1 \leq ES(n) \leq \binom{2n-4}{n-2} + 1 = O(4^n/\sqrt{n}). \]
Theorem (Erdős-Szekeres 1935)

For any positive integers $k$ and $l$, there exists an integer $f(k, l)$, such that any set of at least $f(k, l)$ points in the plane in general position, contains either a $k$-cup or an $l$-cap. Moreover

$$f(k, l) = \binom{k + l - 4}{k - 2} + 1$$
Proof is very combinatorial. The only geometric fact used was the following: Order the points from left to right \( \{p_1, \ldots, p_N\} \)

**transitive property:** If \((p_1, p_2, p_3)\) is a cap (cup), and \((p_2, p_3, p_4)\) is a cap (cup), then \(p_1, p_2, p_3, p_4\) is a 4-cap (4-cup).
Proof is very combinatorial. The only geometric fact used was the following: Order the points from left to right \( \{p_1, ..., p_N\} \)

**transitive property:** If \((p_1, p_2, p_3)\) is a cap (cup), and \((p_2, p_3, p_4)\) is a cap (cup), then \(p_1, p_2, p_3, p_4\) is a 4-cap (4-cup).
Proof is very combinatorial. The only geometric fact used was the following: Order the points from left to right \( \{p_1, \ldots, p_N\} \)

**transitive property:** If \( (p_1, p_2, p_3) \) is a cap (cup), and \( (p_2, p_3, p_4) \) is a cap (cup), then \( p_1, p_2, p_3, p_4 \) is a 4-cap (4-cup).
Proof is very combinatorial. The only geometric fact used was the following: Order the points from left to right \( \{p_1, \ldots, p_N\} \)

**transitive property:** If \((p_1, p_2, p_3)\) is a cap (cup), and \((p_2, p_3, p_4)\) is a cap (cup), then \(p_1, p_2, p_3, p_4\) is a 4-cap (4-cup).
Proof is very combinatorial. The only geometric fact used was the following: Order the points from left to right \( \{p_1, ..., p_N\} \)

**transitive property:** If \((p_1, p_2, p_3)\) is a cap (cup), and \((p_2, p_3, p_4)\) is a cap (cup), then \(p_1, p_2, p_3, p_4\) is a 4-cap (4-cup).
transitive property:
transitive property:
transitive property:
transitive property:
transitive property:
Definition

A family $C$ of convex bodies (compact convex sets) in the plane is said to be in \textit{convex position} if none of its members is contained in the convex hull of the union of the others. We say that $C$ is in \textit{general position} if every three members are in convex position.
Definition

We say that a family of convex bodies in the plane is *noncrossing* if any two members share at most two boundary points.
Theorem (Pach and Tóth 2000)

For any positive integer $n$, there exists an integer $NC(n)$, such that any set of at least $NC(n)$ noncrossing convex bodies in the plane in general position must contain $n$ members in convex position. Moreover

$$2^{n-2} + 1 \leq NC(n) \leq 2^{2^{2^n}}.$$

$NC(n) \leq 2^{2^{cn}}$, Hubard-Montejano-Mora-S. 2011

$NC(n) \leq 2^{c'n^2 \log n}$, Fox-Pach-Sudakov-S. 2012.
Theorem (Pach and Tóth 2000)

For any positive integer $n$, there exists an integer $NC(n)$, such that any set of at least $NC(n)$ noncrossing convex bodies in the plane in general position must contain $n$ members in convex position. Moreover

$$2^{n-2} + 1 \leq NC(n) \leq 2^{2^{2n}}.$$ 

$NC(n) \leq 2^{2^{cn}}$, Hubard-Montejano-Mora-S. 2011

$NC(n) \leq 2^{c'n^2 \log n}$, Fox-Pach-Sudakov-S. 2012.
Finding monochromatic paths in ordered hypergraphs

For an ordered 3-uniform hypergraph \( H = ([N], E) \), a monotone 3-path of length \( n \) are edges
\((v_1, v_2, v_3), (v_2, v_3, v_4), (v_3, v_4, v_5), \ldots, (v_{n-2}, v_{n-1}, v_n)\).

In general, for an ordered \( k \)-uniform hypergraph \( H = ([N], E) \), a monotone \( k \)-path of length \( n \) are edges
\((v_1, v_2, \ldots, v_k), (v_2, v_3, \ldots, v_{k+1}), \ldots, (v_{n-k+1}, \ldots, v_n)\).
Finding monochromatic paths in ordered hypergraphs

For an ordered 3-uniform hypergraph $H = ([N], E)$, a monotone 3-path of length $n$ are edges $(v_1, v_2, v_3), (v_2, v_3, v_4), (v_3, v_4, v_5), \ldots, (v_{n-2}, v_{n-1}, v_n)$.

In general, for an ordered $k$-uniform hypergraph $H = ([N], E)$, a monotone $k$-path of length $n$ are edges $(v_1, v_2, \ldots, v_k), (v_2, v_3, \ldots, v_{k+1}), \ldots, (v_{n-k+1}, \ldots, v_n)$.
Finding monochromatic paths in ordered hypergraphs

For an ordered 3-uniform hypergraph $H = ([N], E)$, a monotone 3-path of length $n$ are edges $(v_1, v_2, v_3), (v_2, v_3, v_4), (v_3, v_4, v_5), \ldots, (v_{n-2}, v_{n-1}, v_n)$.

In general, for an ordered $k$-uniform hypergraph $H = ([N], E)$, a monotone $k$-path of length $n$ are edges $(v_1, v_2, \ldots, v_k), (v_2, v_3, \ldots, v_{k+1}) \ldots, (v_{n-k+1}, \ldots, v_n)$. 

Andrew Suk MIT Geometric Ramsey Theory
For an ordered 3-uniform hypergraph $H = ([N], E)$, a monotone 3-path of length $n$ are edges

$$(v_1, v_2, v_3), (v_2, v_3, v_4), (v_3, v_4, v_5), \ldots, (v_{n-2}, v_{n-1}, v_n).$$

In general, for an ordered $k$-uniform hypergraph $H = ([N], E)$, a monotone $k$-path of length $n$ are edges

$$(v_1, v_2, \ldots, v_k), (v_2, v_3, \ldots, v_{k+1}), \ldots, (v_{n-k+1}, \ldots, v_n).$$
Finding monochromatic paths in ordered hypergraphs

For an ordered 3-uniform hypergraph $H = ([N], E)$, a monotone 3-path of length $n$ are edges $(v_1, v_2, v_3), (v_2, v_3, v_4), (v_3, v_4, v_5), ..., (v_{n-2}, v_{n-1}, v_n)$.

In general, for an ordered $k$-uniform hypergraph $H = ([N], E)$, a monotone $k$-path of length $n$ are edges $(v_1, v_2, ..., v_k), (v_2, v_3, ..., v_{k+1})..., (v_{n-k+1}, ..., v_n)$. 
For an ordered 3-uniform hypergraph $H = ([N], E)$, a monotone 3-path of length $n$ are edges 
$(v_1, v_2, v_3), (v_2, v_3, v_4), (v_3, v_4, v_5), \ldots, (v_{n-2}, v_{n-1}, v_n)$.

In general, for an ordered $k$-uniform hypergraph $H = ([N], E)$, a monotone $k$-path of length $n$ are edges 
$(v_1, v_2, \ldots, v_k), (v_2, v_3, \ldots, v_{k+1})\ldots, (v_{n-k+1}, \ldots, v_n)$.
Ordered hypergraphs

**Definition**

Let $N_k(q, n)$ denote the smallest integer $N$ such that for every $q$ coloring on the $k$-tuples of the set $[N]$ contains a monochromatic monotone $k$-path of length $n$.

\[ N_2(q, n) = (n - 1)^q + 1 \] by Dilworth's theorem.
Definition

Let $N_k(q, n)$ denote the smallest integer $N$ such that for every $q$ coloring on the $k$-tuples of the set $[N]$ contains a monochromatic monotone $k$-path of length $n$.

$$N_2(q, n) = (n - 1)^q + 1$$ by Dilworth’s theorem.
Definition

Let $N_k(q, n)$ denote the smallest integer $N$ such that for every $q$ coloring on the $k$-tuples of the set $[N]$ contains a monochromatic monotone $k$-path of length $n$.

$$N_2(q, n) = (n - 1)^q + 1 \text{ by Dilworth’s theorem.}$$
Ordered hypergraphs

**Definition**

Let $N_k(q, n)$ denote the smallest integer $N$ such that for every $q$ coloring on the $k$-tuples of the set $[N]$ contains a monochromatic monotone $k$-path of length $n$.

$$N_2(q, n) = (n - 1)^q + 1$$ by Dilworth’s theorem.
$N_3(2, n) = \binom{2n-4}{n-2} + 1$ by the Erdős-Szekeres cups-caps (red-blue) argument.
$N_3(2, n) = \binom{2n-4}{n-2} + 1$ by the Erdős-Szekeres cups-caps (red-blue) argument.
$N_3(2, n) = \binom{2n-4}{n-2} + 1$ by the Erdős-Szekeres cups-caps (red-blue) argument.
\[ N_3(2, n) = \binom{2n-4}{n-2} + 1 \] by the Erdős-Szekeres cups-caps (red-blue) argument.
$N_3(2, n) = \binom{2n-4}{n-2} + 1$ by the Erdős-Szekeres cups-caps (red-blue) argument.
$N_3(2, n) = \binom{2n-4}{n-2} + 1$ by the Erdős-Szekeres cups-caps (red-blue) argument.
$N_3(2, n) = \binom{2n-4}{n-2} + 1$ by the Erdős-Szekeres cups-caps (red-blue) argument.
$N_3(2, n) = \binom{2n-4}{n-2} + 1$ by the Erdős-Szekeres cups-caps (red-blue) argument.
\[ N_3(2, n) = \binom{2n-4}{n-2} + 1 \] by the Erdős-Szekeres cups-caps (red-blue) argument.
For more colors.

Theorem (Fox, Pach, Sudakov, S. 2012)

For $q \geq 3$, we have

$$2^{(n/q)^{q-1}} \leq N_3(q, n) \leq 2^{n^{q-1} \log n},$$

Application: Noncrossing convex bodies problem, $NC(n)$.
Obtain the **transitive property** on triples of convex bodies.

$$NC(n) \leq N_3(3, n) \leq 2^{n^2 \log n}$$
Proof of $N_3(q, n) \leq N_2(n^{q-1}, n) \leq n^{n^{q-1}} = 2^{n^{q-1} \log n}$:

1. Set $N = N_2(n^{q-1}, n)$

2. $\chi : \binom{[N]}{3} \to [q]$ be $q$-coloring on the triples of $[N]$.

3. Then define $\phi : \binom{[N]}{2} \to [n]^{q-1}$ as follows. We color $(i, j) \in \binom{[N]}{2}$ with color $(a_1, a_2, \ldots, a_{q-1})$ where $a_t$ denotes the length of the longest monotone 3-path ending with vertices $i, j$ in color $t$. 

![Diagram](image-url)
Proof of $N_3(q, n) \leq N_2(n^{q-1}, n) \leq n^{n^{q-1}} = 2^{n^{q-1} \log n}$:

1. Set $N = N_2(n^{q-1}, n)$
2. $\chi : \binom{[N]}{3} \to [q]$ be $q$-coloring on the triples of $[N]$.
3. Then define $\phi : \binom{[N]}{2} \to [n]^{q-1}$ as follows. We color $(i, j) \in \binom{[N]}{2}$ with color $(a_1, a_2, ..., a_{q-1})$ where $a_t$ denotes the length of the longest monotone 3-path ending with vertices $i, j$ in color $t$. 

[Diagram of a line with points labeled 1 to N, with a circle highlighting a segment between points 2 and 3.]
**Proof of** $N_3(q, n) \leq N_2(n^{q-1}, n) \leq n^{n^{q-1}} = 2^{n^{q-1} \log n}$:

1. Set $N = N_2(n^{q-1}, n)$
2. $\chi : \binom{[N]}{3} \to [q]$ be $q$-coloring on the triples of $[N]$.
3. Then define $\phi : \binom{[N]}{2} \to [n]^{q-1}$ as follows. We color $(i, j) \in \binom{[N]}{2}$ with color $(a_1, a_2, \ldots, a_{q-1})$ where $a_t$ denotes the length of the longest monotone 3-path ending with vertices $i, j$ in color $t$. 

![Diagram](image-url)
Proof of $N_3(q, n) \leq N_2(n^{q-1}, n) \leq n^{n^{q-1}} = 2^{n^{q-1} \log n}$:

1. Set $N = N_2(n^{q-1}, n)$
2. $\chi : \binom{[N]}{3} \to [q]$ be $q$-coloring on the triples of $[N]$.
3. Then define $\phi : \binom{[N]}{2} \to [n]^{q-1}$ as follows. We color $(i, j) \in \binom{[N]}{2}$ with color $(a_1, a_2, \ldots, a_{q-1})$ where $a_t$ denotes the length of the longest monotone 3-path ending with vertices $i, j$ in color $t$. 

![Diagram showing a line of points labeled 1 to N with a circle around a subset of points 4 to 6, indicating the subset of interest.](image)
Proof of \( N_3(q, n) \leq N_2(n^{q-1}, n) \leq n^{n^{q-1}} = 2^{n^{q-1} \log n} \):

1. Set \( N = N_2(n^{q-1}, n) \)

2. \( \chi: \binom{[N]}{3} \rightarrow [q] \) be \( q \)-coloring on the triples of \([N]\).

3. Then define \( \phi: \binom{[N]}{2} \rightarrow [n]^{q-1} \) as follows. We color \((i, j) \in \binom{[N]}{2}\) with color \((a_1, a_2, \ldots, a_{q-1})\) where \(a_t\) denotes the length of the longest monotone 3-path ending with vertices \(i, j\) in color \(t\).
Proof of $N_3(q, n) \leq N_2(n^{q-1}, n) \leq n^{n^{q-1}} = 2^{n^{q-1}\log n}$:

1. Set $N = N_2(n^{q-1}, n)$
2. $\chi : \binom{[N]}{3} \to [q]$ be $q$-coloring on the triples of $[N]$.
3. Then define $\phi : \binom{[N]}{2} \to [n]^{q-1}$ as follows. We color $(i, j) \in \binom{[N]}{2}$ with color $(a_1, a_2, \ldots, a_{q-1})$ where $a_t$ denotes the length of the longest monotone 3-path ending with vertices $i, j$ in color $t$. 

![Diagram](image-url)
Proof of $N_3(q, n) \leq N_2(n^{q-1}, n) \leq n^{n^{q-1}} = 2^{n^{q-1}\log n}$:

1. Set $N = N_2(n^{q-1}, n)$
2. $\chi : \binom{[N]}{3} \to [q]$ be $q$-coloring on the triples of $[N]$.
3. Then define $\phi : \binom{[N]}{2} \to [n]^{q-1}$ as follows. We color $(i, j) \in \binom{[N]}{2}$ with color $(a_1, a_2, \ldots, a_{q-1})$ where $a_t$ denotes the length of the longest monotone 3-path ending with vertices $i, j$ in color $t$. 

andrew suk mit geometric ramsey theory
Proof of $N_3(q, n) \leq N_2(n^{q-1}, n) \leq n^{n^{q-1}} = 2^{n^{q-1} \log n}$:

1. Set $N = N_2(n^{q-1}, n)$

2. $\chi : \binom{[N]}{3} \rightarrow [q]$ be $q$-coloring on the triples of $[N]$.

3. Then define $\phi : \binom{[N]}{2} \rightarrow [n]^{q-1}$ as follows. We color $(i, j) \in \binom{[N]}{2}$ with color $(a_1, a_2, \ldots, a_{q-1})$ where $a_t$ denotes the length of the longest monotone 3-path ending with vertices $i, j$ in color $t$. 

![Diagram of colored paths]
Proof of $N_3(q, n) \leq N_2(n^{q-1}, n) \leq n^{n^{q-1}} = 2^{n^{q-1}\log n}$:

1. Set $N = N_2(n^{q-1}, n)$

2. $\chi : \binom{[N]}{3} \to [q]$ be $q$-coloring on the triples of $[N]$.

3. Then define $\phi : \binom{[N]}{2} \to [n]^{q-1}$ as follows. We color $(i, j) \in \binom{[N]}{2}$ with color $(a_1, a_2, ..., a_{q-1})$ where $a_t$ denotes the length of the longest monotone 3-path ending with vertices $i, j$ in color $t$. 

![Diagram](image-url)
Proof of $N_3(q, n) \leq N_2(n^{q-1}, n) \leq n^{n^{q-1}} = 2^{n^{q-1} \log n}$:

1. Set $N = N_2(n^{q-1}, n)$
2. $\chi : \binom{[N]}{3} \to [q]$ be $q$-coloring on the triples of $[N]$.
3. Then define $\phi : \binom{[N]}{2} \to [n]^{q-1}$ as follows. We color $(i, j) \in \binom{[N]}{2}$ with color $(a_1, a_2, \ldots, a_{q-1})$ where $a_t$ denotes the length of the longest monotone 3-path ending with vertices $i, j$ in color $t$.

![Graph](image.png)
Proof of $N_3(q,n) \leq N_2(n^{q-1},n) \leq n^{n^{q-1}} = 2^{n^{q-1} \log n}$:

1. Set $N = N_2(n^{q-1},n)$

2. $\chi : \binom{[N]}{3} \rightarrow [q]$ be $q$-coloring on the triples of $[N]$.

3. Then define $\phi : \binom{[N]}{2} \rightarrow [n]^{q-1}$ as follows. We color $(i,j) \in \binom{[N]}{2}$ with color $(a_1, a_2, \ldots, a_{q-1})$ where $a_t$ denotes the length of the longest monotone 3-path ending with vertices $i,j$ in color $t$. 

---

Andrew Suk MIT  
Geometric Ramsey Theory
Proof of $N_3(q, n) \leq N_2(n^{q-1}, n) \leq n^{n^{q-1}} = 2^{n^{q-1}\log n}$:

1. Set $N = N_2(n^{q-1}, n)$

2. $\chi : ([N]^3) \rightarrow [q]$ be $q$-coloring on the triples of $[N]$.

3. Then define $\phi : ([N]^2) \rightarrow [n]^{q-1}$ as follows. We color $(i,j) \in ([N]^2)$ with color $(a_1, a_2, \ldots, a_{q-1})$ where $a_t$ denotes the length of the longest monotone 3-path ending with vertices $i,j$ in color $t$. 

![Diagram](image-url)
By definition of $N_2(n^{q-1}, n)$, there is monochromatic 2-path on vertices $v_1 < v_2 < \ldots < v_n$ with color $(a_1^*, \ldots, a_{q-1}^*)$. 

\[ (a_1^*, a_2^*, \ldots, a_{q-1}^*) \] 

\[ (a_1^*, a_2^*, \ldots, a_{q-1}^*) \] 

\[ (a_1^*, a_2^*, \ldots, a_{q-1}^*) \] 

\[ v_1 \longrightarrow v_2 \longrightarrow v_3 \quad \ldots \ldots \quad v_{n-1} \longrightarrow v_n \]
Claim: $(v_1, \ldots, v_n)$ is a monochromatic 3-path (with color $q$)!
Indeed, Assume $(v_i, v_{i+1}, v_{i+2})$ has color $j \neq q$.

1. Longest $j$th-colored 3-path ending with vertices $(v_i, v_j)$ must be shorter than the longest $j$th-colored 3-path ending with vertices $(v_{j+1}, v_{j+2})$.
2. Contradicts $\phi(v_i, v_{i+1}) = \phi(v_{i+1}, v_{i+2})$.
3. Hence $(v_i, v_{i+1}, v_{i+2})$ must have color $q$ for all $i$. 

$(a_1^*, \ldots, a_j^*, \ldots, a_{q-1}^*)$
**Claim:** \((v_1, ..., v_n)\) is a monochromatic 3-path (with color \(q\))!

Indeed, Assume \((v_i, v_{i+1}, v_{i+2})\) has color \(j \neq q\).

1. Longest \(j\)th-colored 3-path ending with vertices \((v_i, v_j)\) must be shorter than the longest \(j\)th-colored 3-path ending with vertices \((v_{j+1}, v_{j+2})\).
2. Contradicts \(\phi(v_i, v_{i+1}) = \phi(v_{i+1}, v_{i+2})\).
3. Hence \((v_i, v_{i+1}, v_{i+2})\) must have color \(q\) for all \(i\).
The upper bound proof can easily be generalized to show

\[ N_k(q, n) \leq N_{k-1}((n - k + 1)^{q-1}, n) \]

Using the stepping-up approach we have

**Theorem (Fox-Pach-Sudakov-S. 2012)**

Define \( t_1(x) = x \) and \( t_{i+1}(x) = 2^{t_i(x)} \). Then for \( k \geq 4 \) we have

\[ t_{k-1}(cn^{q-1}) \leq N_k(q, n) \leq t_{k-1}(c'n^{q-1} \log n). \]

Recall Ramsey numbers:

\[ t_{k-1}(cn^2) \leq R_k(n) \leq t_k(c'n). \]
For $k \geq 3$ we have

\[ t_{k-1}(cn^{q-1}) \leq N_k(q, n) \leq t_{k-1}(c' n^{q-1}). \]

Noncrossing convex bodies problems:

\[ NC(n) \leq 2^{n^2 \log n} \Rightarrow NC(n) \leq 2^{n^2} \]

For $k \geq 3$,

\[ t_{k-1}(cn^2) \leq R_k(n) \leq t_k(c'n). \]
Combinatorial Problem

\[ 2^{cn^2} \leq R_3(n) \leq 2^{2c'n}. \]

**Problem**

*Close the gap on \( R_3(n) \)*

**Conjecture (Erdős, $500 problem)**

\[ 2^{2cn} \leq R_3(n) \]

**Erdős-Hajnal Stepping Up Lemma:** \( x < R_k(n) \), then \( 2^x \lesssim R_{k+1}(n) \) for \( k \geq 3 \)

Would imply \( R_4(n) = 2^{2^{2\Theta(n)}} \), and \( R_k(n) = t_k(\Theta(n)) \).

Is there a geometric construction showing \( 2^{2cn} \leq R_3(n) \)?
\( V = \{ N \text{ points in the plane in general position} \} \\
E = \{ \text{triples with a clockwise orientation} \} \\

Many graphs and hypergraphs defined geometrically.
$V = \{N \text{ tubes of length } l \text{ and radius } 1 \text{ in } \mathbb{R}^d\}$
$E = \{\text{pairs that intersect}\}.$

Semi-algebraic hypergraphs.
A set $A \subset \mathbb{R}^d$ is called semi-algebraic if there are polynomials $f_1, f_2, \ldots, f_r \in \mathbb{R}[x_1, \ldots, x_d]$ and a Boolean formula $\Phi(X_1, X_2, \ldots, X_r)$, where $X_1, \ldots, X_r$ are variables attaining values “true” and “false”, such that

$$A = \left\{ x \in \mathbb{R}^d : \Phi(f_1(x) \geq 0, \ldots, f_r(x) \geq 0) \right\}.$$

$\Phi$ involves unions, intersections, and complementations. Assume Quantifier-free (Tarski’s Theorem).

$A$ has complexity at most $t$ if $d, r \leq t$ and each $\deg(f_i) \leq t$.

Examples: hyperplanes, balls, boxes, tubes, etc. in $\mathbb{R}^d$. 
Encode sets to points

Let \( V = \{ A_1, ..., A_N \} \) be a family of \( N \) semi-algebraic sets in \( \mathbb{R}^d \), each set with complexity at most \( t \).

\[
A_i = \left\{ x \in \mathbb{R}^d : \Phi(f_1(x) \geq 0, ..., f_r(x) \geq 0) \right\}.
\]

**Encode each set:** \( A_i \rightarrow p_i \in \mathbb{R}^q \) for \( q = q(t) \).

\( V = \{ p_1, ..., p_N \} \), \( N \) points in \( \mathbb{R}^q \).
Encode sets to points

Let $V = \{A_1, ..., A_N\}$ be a family of $N$ semi-algebraic sets in $\mathbb{R}^d$, each set with complexity at most $t$.

$$A_i = \left\{ x \in \mathbb{R}^d : \Phi (f_1(x) \geq 0, ..., f_r(x) \geq 0) \right\} .$$

**Encode each set:** $A_i \rightarrow p_i \in \mathbb{R}^q$ for $q = q(t)$.

$V = \{p_1, ..., p_N\}$, $N$ points in $\mathbb{R}^q$. 
Encode sets to points

Let $V = \{A_1, \ldots, A_N\}$ be a family of $N$ semi-algebraic sets in $\mathbb{R}^d$, each set with complexity at most $t$.

$$A_i = \left\{ x \in \mathbb{R}^d : \Phi(f_1(x) \geq 0, \ldots, f_r(x) \geq 0) \right\}.$$  

**Encode each set:** $A_i \rightarrow p_i \in \mathbb{R}^q$ for $q = q(t)$.

$V = \{p_1, \ldots, p_N\}$, $N$ points in $\mathbb{R}^q$.  

Andrew Suk MIT Geometric Ramsey Theory
Encode sets to points

Let $V = \{A_1, ..., A_N\}$ be a family of $N$ semi-algebraic sets in $\mathbb{R}^d$, each set with complexity at most $t$.

$$A_i = \left\{ x \in \mathbb{R}^d : \Phi(f_1(x) \geq 0, ..., f_r(x) \geq 0) \right\}.$$  

Encode each set: $A_i \rightarrow p_i \in \mathbb{R}^q$ for $q = q(t)$.

$V = \{p_1, ..., p_N\}$, $N$ points in $\mathbb{R}^q$. 
Encode sets to points

Let $V = \{A_1, \ldots, A_N\}$ be a family of $N$ semi-algebraic sets in $\mathbb{R}^d$, each set with complexity at most $t$.

$A_i = \left\{ x \in \mathbb{R}^d : \Phi (f_1(x) \geq 0, \ldots, f_r(x) \geq 0) \right\}$.

**Encode each set:** $A_i \to p_i \in \mathbb{R}^q$ for $q = q(t)$.

$V = \{p_1, \ldots, p_N\}$, $N$ points in $\mathbb{R}^q$. 
For $V = \{p_1, ..., p_N\} \subset \mathbb{R}^q$, the edge set $E \subset \binom{V}{k}$ is semi-algebraic if $E$ can be described with a constant number of polynomial equations and inequalities (each of bounded degree), and a boolean formula $\Phi$. 
Semi-algebraic relation

For $V = \{p_1, ..., p_N\} \subset \mathbb{R}^q$, the edge set $E \subset \binom{V}{k}$ is semi-algebraic if there exists a semi-algebraic set $E^* \subset \mathbb{R}^{kq}$ with bounded description complexity, such that for $i_1 < \cdots < i_k$

$$(p_{i_1}, ..., p_{i_k}) \in E \iff (p_{i_1}, ..., p_{i_k}) \in E^* \subset \mathbb{R}^{kq}.$$  

Example: For $k = 3$ look at all triples $(p_{i_1}, p_{i_2}, p_{i_3})$ in $\mathbb{R}^{3q}$. Call the pair $(V, E)$ a semi-algebraic $k$-uniform hypergraph (with bounded description complexity).
\[ V = \{ A_1, \ldots, A_N \}, \] 
\[ N \text{ disks in the plane. } E = \{ \text{pairs of disks that intersect} \}. \]
$V = \{A_1, ..., A_N\}$, $N$ disks in the plane. $E = \{\text{pairs of disks that intersect}\}$.
\[ V = \{A_1, \ldots, A_N\}, \ N \text{ disks in the plane.} \ E = \{\text{pairs of disks that intersect}\}. \]
$V = \{A_1, \ldots, A_N\}$, $N$ disks in the plane. $E = \{\text{pairs of disks that intersect}\}$.

$A_i \rightarrow p_i = (x_i, y_i, r_i)$, $A_j \rightarrow p_j = (x_j, y_j, r_j)$. $A_i$ and $A_j$ cross if and only if

$$-x_i^2 + 2x_ix_j - x_j^2 - y_i^2 + 2y_iy_j - y_j^2 + r_i^2 + 2r_ir_j + r_j^2 \geq 0.$$
\[ V = \{A_1, \ldots, A_N\}, \text{ } N \text{ disks in the plane. } E = \{ \text{ pairs of disks that intersect} \}. \]

\[ (V, E) \text{ is semi-algebraic graph, } \]

\[ E^* = \{(z_1, \ldots, z_6) \in \mathbb{R}^6 : f(z_1, \ldots, z_6) \geq 0\}, \text{ where } \]

\[ f(z_1, \ldots, z_6) = -z_1^2 + 2z_1z_4 - z_4^2 - z_2^2 + 2z_2z_5 - z_5^2 + z_3^2 + 2z_3z_6 + z_6^2. \]

\[ (p_i, p_j) \in E \iff (p_i, p_j) \in E^*. \]
Examples

1. $V = \{N \text{ circles in } \mathbb{R}^3\}$
   $E = \{\text{pairs that are linked}\}$.

2. $V = \{N \text{ hyperplanes in } \mathbb{R}^d \text{ in general position}\}$,
   $E = \{d\text{-tuples whose intersection point is above the hyperplane } x_d = 0\}$. 

\[ \begin{array}{c}
\text{Diagram of circles in } \mathbb{R}^3 \\
\text{Diagram of hyperplanes in } \mathbb{R}^d \\
x_d = 0
\end{array} \]
**Definition:** Let $R_{k}^{\text{semi}}(n)$ be the minimum integer $N$ such that any $N$-vertex semi-algebraic $k$-uniform hypergraph $H = (V, E)$ contains either a clique or an independent set of size $n$. $R_{k}^{\text{semi}}(n) \leq R_{k}(n)$.

**Theorem (Alon, Pach, Pinchasi, Radoičić, Sharir 2005)**

$$R_{2}^{\text{semi}}(n) \leq n^{c_1}.$$ 

Applying Milnor-Thom Theorem and Cutting Lemma:

**Theorem (Conlon, Fox, Pach, Sudakov, S. 2012)**

For $k \geq 3$,

$$t_{k-1}(c_2 n) \leq R_{k}^{\text{semi}}(n) \leq t_{k-1}(n^{c_1}).$$

**Recall:** for $k \geq 3$, $t_{k-1}(cn^2) \leq R_{k}(n) \leq t_{k}(c'n)$. 
Theorem (Conlon, Fox, Pach, Sudakov, S. 2012)

for $k \geq 3$,

$$t_{k-1}(c_2 n) \leq R_{k}^{semi}(n) \leq t_{k-1}(n^{c_1}).$$

Several applications...

Determine the minimum integer $OSH_d(n)$, such that any family of at least $OSH_d(n)$ hyperplanes in $\mathbb{R}^d$ in general position, must contain $n$ members such that every $d$-tuple intersects on one-side of the hyperplane $x_d = 0$.

$$OSH_2(n) = \Theta(n^2), \quad OSH_d(n) \leq R_d(n) \leq t_d(c'n).$$

$V = \{N \text{ hyperplanes}\}$,  
$E = \{d\text{-tuples that intersect above } x_d = 0 \text{ hyperplane}\}$.  
**New bound:** $OSH_d(n) \leq R_{d}^{semi}(n) \leq t_{d-1}(n^{c_1})$
Ramsey number of 3-uniform hypergraphs.

\[ 2^{cn^2} \leq R_3(n) \leq 2^{2^{c'n}}. \]

**Conjecture (Erdős)**

\[ 2^{2^{cn}} \leq R_3(n) \]

Is there a geometric construction showing \( 2^{2^{cn}} \leq R_3(n) \)?

**Our Result:** \( R_3^{semi}(n) \leq 2^{n^{c_1}} \).
Regularity lemma for semi-algebraic graphs (and hypergraphs).

**Lemma (Regularity Lemma, Szemerédi)**

Let $G = (V, E)$ be an $N$-vertex graph with $\epsilon N^2$ edges. Then there exists a partition $V = \{V_1, \ldots, V_M\}$ into $M(\epsilon)$ equal parts, such that all but at most $\epsilon \binom{M}{2}$ pairs of parts are regular.
Future work

Regularity lemma for semi-algebraic graphs (and hypergraphs).

Lemma (Regularity Lemma, Szemerédi)

Let $G = (V, E)$ be an $N$-vertex graph with $\epsilon N^2$ edges. Then there exists a partition $V = \{V_1, ..., V_M\}$ into $M(\epsilon)$ equal parts, such that all but at most $\epsilon \binom{M}{2}$ pairs of parts are regular.
Future work

Regularity lemma for semi-algebraic graphs (and hypergraphs).

Lemma (Regularity Lemma, Szemerédi)

Let $G = (V, E)$ be an $N$-vertex graph with $\epsilon N^2$ edges. Then there exists a partition $V = \{V_1, \ldots, V_M\}$ into $M(\epsilon)$ equal parts, such that all but at most $\epsilon \binom{M}{2}$ pairs of parts are regular.
Future work

Regularity lemma for semi-algebraic graphs (and hypergraphs).

**Lemma (Regularity Lemma, Szemerédi)**

Let $G = (V, E)$ be an $N$-vertex graph with $\epsilon N^2$ edges. Then there exists a partition $V = \{V_1, ..., V_M\}$ into $M(\epsilon)$ equal parts, such that all but at most $\epsilon \binom{M}{2}$ pairs of parts are regular.

**Szemerédi:** $M(\epsilon) \leq \frac{1}{\epsilon^5} (2)$.

**Semi-algebraic graphs:**

1. regular $\xRightarrow{?} \text{ complete or empty}.$

2. $M(\epsilon) \xRightarrow{?} \frac{1}{\epsilon^c}.$

Attack other problems in discrete geometry in a semi-algebraic setting.
Future work

Unit distance problem in $\mathbb{R}^2$ and $\mathbb{R}^3$.

**Conjecture (Erdős, $500$)**

*Given $N$ points in the plane, no more than $N^{1+c/\log \log N}$ pairs can be unit distance apart.*

$V = \{N \text{ points in the plane}\}$

$E = \{\text{Pairs that are a unit distance apart}\}$
Thank you!
1. **Milnor-Thom theorem**: $M$ bounded degree surfaces partitions $\mathbb{R}^q$ into $O(M^q)$ cells.

2. **Cutting lemma** (Chazelle, Edelsbrunner, Guibas, Sharir): Given $M$ bounded degree surfaces $\Sigma$ in $\mathbb{R}^q$ and integer $r$, we can partition $\mathbb{R}^q$ into $O(r^{2q})$ “simple” regions (cells) such that each cell is “crossed” by $O(M/r)$ surfaces from $\Sigma$. 
1. **Milnor-Thom theorem**: $M$ bounded degree surfaces partitions $\mathbb{R}^q$ into $O(M^q)$ cells.

2. **Cutting lemma** (Chazelle, Edelsbrunner, Guibas, Sharir): Given $M$ bounded degree surfaces $\Sigma$ in $\mathbb{R}^q$ and integer $r$, we can partition $\mathbb{R}^q$ into $O(r^{2q})$ “simple” regions (cells) such that each cell is “crossed” by $O(M/r)$ surfaces from $\Sigma$. 

![Diagram](image-url)
1. **Milnor-Thom theorem**: $M$ bounded degree surfaces partitions $\mathbb{R}^q$ into $O(M^q)$ cells.

2. **Cutting lemma** (Chazelle, Edelsbrunner, Guibas, Sharir): Given $M$ bounded degree surfaces $\Sigma$ in $\mathbb{R}^q$ and integer $r$, we can partition $\mathbb{R}^q$ into $O(r^{2q})$ “simple” regions (cells) such that each cell is “crossed” by $O(M/r)$ surfaces from $\Sigma$. 
Getting an exponential improvement.

\[ R_{k}^{semi}(n) \leq t_{k-1}(n^c). \]

Combining a combinatorial argument + the Milnor-Thom theorem + cutting lemma,

**Theorem (Conlon, Fox, Pach, Sudakov, S. 2012)**

\[ R_{k+1}^{semi}(n) \leq 2\tilde{O}(R_{k}^{semi}(n)) \]

\[ R_{2}^{semi}(n) \leq n^c \quad R_{3}^{semi}(n) \leq 2^{n^{c_1}} \]

\[ R_{4}^{semi}(n) \leq 2^{2^{n^{c_1}}}, \ldots \]
Semi-algebraic $k$-uniform hypergraph $H = (V, E)$, $V = \{p_1, ..., p_N\} \subset \mathbb{R}^q,$

$E^* = \{(x_1, ..., x_k) \subset \mathbb{R}^{kq} : \Phi(f_1(x_1, ..., x_k) > 0, ..., f_t(x_1, ..., x_k) > 0)\}$

Every $k-1$-tuple of points, $p_{i_1}, ..., p_{i_{k-1}},$ gives rise to $t$ bounded degree surfaces in $\mathbb{R}^q$.

\[
\{ f_1(p_{i_1}, ..., p_{i_{k-1}}, x_k) = 0 \}, ..., \{ f_t(p_{i_1}, ..., p_{i_{k-1}}, x_k) = 0 \} \subset \mathbb{R}^q.
\]
$(p_{i_1}, p_{i_2}, ..., p_{i_k}) \in E$?

Sign pattern $(f_1(p_{i_1}, p_{i_2}, ..., p_{i_k}), ..., f_t(p_{i_1}, p_{i_2}, ..., p_{i_k}))$. I.e. $(+,-,+,0,+,+)$.

$$E^* = \{(x_1, ..., x_k) \in \mathbb{R}^{2q} : \Phi(f_1(x_1, ..., x_k) > 0, ..., f_t(x_1, ..., x_k) > 0)\}$$
Our problem is about: $N$ points in $\mathbb{R}^q$ and $M = t\left(\binom{N}{k-1}\right)$ bounded degree surfaces in $\mathbb{R}^q$. 