1. The coinvariant algebra is \( C_n := \mathbb{C}[x_1,\ldots,x_n]/I_n \), where \( I_n = \langle e_1,\ldots,e_n \rangle \) is the ideal generated by the elementary symmetric polynomials \( e_i = e_i(x_1,\ldots,x_n), i = 1,\ldots,n \). Show that \( C_n \) is an \( n! \)-dimensional vector space with a linear basis given by the cosets of the monomials \( x_1^{a_1}\cdots x_{n-1}^{a_{n-1}} \) with \( a_i \leq n-i \) for all \( i \).

2. Find the number of permutations \( w \in S_n \) such that any two reduced decompositions for \( w \) can be related to each other by a sequence of 2-moves.

3. Let \( w \) be any permutation in \( S_n \) and \( w_0 \) be the longest permutation in \( S_n \). Show that \( \mathcal{S}_w(x_1,\ldots,x_n) \equiv \mathcal{S}_{w_0 \cdot w_0}(-x_n,\ldots,-x_1) \mod I_n \) from problem 1.

4. Let \( h_i(x) := 1+x u_i, i = 1,\ldots,n-1 \), where the \( u_i \) are the generators of the nilHecke algebra and \( x \) commutes with all \( u_i \)'s. The \( h_i(x) \) satisfy the Yang-Baxter relations. Let

\[
\phi_n(x,y) := \prod_{i=1}^{n-1} \prod_{j=n-i}^{n} h_{i+j-1}(x_i - y_j).
\]

Use the Yang-Baxter relations to show that \( \phi_n(x,y) = \phi_n(0,y) \cdot \phi_n(x,0) \).

5. (A) In class we constructed the insertion procedure for RC-graphs. Show that it is well-defined and invertible. Deduce Monk’s rule.

(B) Show that, in case of RC-graphs for Grassmannian permutations, this procedure is equivalent to the RSK insertion for SSYTs.

6. Let us say that a permutation \( w \in S_n \) is strictly dominant if its code \( \text{code}(w) = (c_1,c_2,\ldots,c_n) \) is a strict partition, that is \( c_1 > c_2 > \cdots > c_k = \cdots = c_n = 0 \).

(A) Show that the following conditions are equivalent:

1. \( w \) is strictly dominant.
2. \( w w_0 \) is strictly dominant.
3. \( w \) is of the form \( w_1 > w_2 > \cdots > w_k < w_{k+1} < \cdots < w_n \).
4. \( w \) is both 132-avoiding and 231-avoiding.

(B) Find the number of strictly dominant permutations in \( S_n \).

7. The Schubert-Kostka matrix is the \( n! \times n! \) matrix \( K = (K_{w,a}) \) defined by \( \mathcal{S}_w(x) = \sum_a K_{w,a} x^a \), for \( w \in S_n \). In other words, \( K_{w,a} \)
counts the number of RC-graphs for $w$ of $x$-weight equal to $x^a$. Let $K^{-1} = (K^{-1}_{a,w})$ be the inverse matrix, that is $\sum_a K_{a,a} K^{-1}_{a,w} = \delta_{u,w}$ and $\sum_w K^{-1}_{a,w} K_{w,b} = \delta_{a,b}$.

(A) Let $w$ be a strictly dominant permutation in $S_n$ with code $(c_1 > c_2 > \cdots > c_k = \cdots = 0)$. Assume that $a = (a_1, \ldots, a_k, 0, \ldots, 0)$. Prove that

$$K^{-1}_{a,w} = \begin{cases} (-1)^{\ell(\sigma)} & \text{if } (a_1, \ldots, a_k) = (c_{\sigma_1}, \ldots, c_{\sigma_k}) \text{ for some } \sigma \in S_k, \\ 0 & \text{otherwise.} \end{cases}$$

(B) Now assume that $w$ is any 312-avoiding permutation. Prove that in this case $K^{-1}_{a,w}$ also equals 0, 1, or $-1$. Find the exact value of $K^{-1}_{a,w}$ in this case.

(C) Is it always true that $K^{-1}_{a,w} \in \{1, -1, 0\}$?