PROBLEM SET 4  (due on Tuesday 11/09/2004)

The problems worth 10 points each.

Problem 1 Let $a_n$ be the sequence of integers given by the recurrence relation $a_n = 2a_{n-1} + 3a_{n-2}$, for $n \geq 2$, and $a_0 = a_1 = 2$. Find an explicit formula for the numbers $a_n$.

Problem 2 Find two constants $b$ and $c$ such that the sequence $a_n = 10^n - 2^n$ satisfies the recurrence relation $a_n = b a_{n-1} + c a_{n-2}$.

Problem 3 Let $a_n$ be the sequence given by the recurrence relation $a_n = a_{n-2} + 1$, for $n \geq 2$, and $a_0 = a_1 = 1$. Find the ordinary and the exponential generating functions of this sequence. Can you give a combinatorial interpretation of the numbers $a_n$ in terms of partitions of some kind?

Problem 4 Two sequences $\{a_n\}_{n \geq 0}$ and $\{b_n\}_{n \geq 0}$ are related by $b_n = \sum_{k=0}^{n} a_k$. What is the relationship between ordinary generating functions of these sequences?

Problem 5 Two sequences $\{a_n\}_{n \geq 0}$ and $\{b_n\}_{n \geq 0}$ are related by $b_n = \sum_{k \geq 0}^{\infty} \frac{a_{n+k}}{k!}$. (Assume that the sum converges.) What is the relationship between exponential generating functions of these sequences?

Problem 6 A star is a graph that contains a vertex adjacent to all edges of the graph. (In particular, the graph with a single vertex and the graph with a single edge are stars.) Let us say that a simple graph $G$ is stellar if every connected component of $G$ is a star. Let $s_n$ be the number of stellar subgraphs of the complete graph $K_n$. We have $s_0 = 1$, $s_1 = 1$, $s_2 = 2$, $s_3 = 7$, . . . . Find the exponential generating function $\sum_{n \geq 0} s_n x^n n!$ for the numbers $s_n$.

Problem 7 Let $C_n = \frac{1}{n+1} \binom{2n}{n}$ be the Catalan number and $C(x) = \sum_{n \geq 0} C_n x^n$. The Motzkin number $M_n$ is defined as the number of paths from $(0, 0)$ to $(n, 0)$ that never go below the $x$-axis and are made of the steps $(1, 0)$, $(1, 1)$, and $(1, -1)$. We have $M_0 = 1$, $M_1 = 1$, $M_2 = 2$, $M_3 = 4$, . . . . Let $M(x) = 1 + \sum_{n \geq 0} M_n x^{n+1} = 1 + x + x^2 + 2x^3 + 4x^4 + \cdots$. Show that $M(x) = C(x/(1+x))$.

Problem 8 Calculate the determinant of the almost upper-triangular $n \times n$-matrix

$$A = (a_{ij}), \quad a_{ij} = \begin{cases} 2^{j-i+1} & \text{if } j - i + 1 \geq 0; \\ \text{otherwise}. \end{cases}$$

Problem 9 Find a bijection between (unlabelled) plane binary trees with $n$ leaves and (unlabelled) planar rooted trees with $n$ vertices. (The numbers of these objects are the Catalan numbers.)
Problem 10 For a Catalan path $P$, let $h(P)$ be the number points in $P$ located on the $x$-axis (excluding the initial and the final points $(0, 0)$ and $(2n, 0)$). Let $C_n(q)$ be the sum $\sum_P q^{h(P)}$ over Catalan paths of length $2n$. For example, $C_2(q) = 1 + q$ and $C_3(q) = 2 + 2q + q^2$. In particular, $C_n(1)$ is the Catalan number. Find an explicit expression for the generating function $C(x, q) = \sum_{n \geq 0} C_n(q) x^n = 1 + x + (1 + q)x^2 + (2 + 2q + q^2)x^3 + \cdots$ for these polynomials. Hint: Try to express $C(x, q)$ in terms of the generating function for the Catalan numbers.

Bonus Problems

Problem 11 (*) Calculate the determinant of the 5-diagonal $n \times n$-matrix $A = (a_{ij})$, where $a_{ij} = \begin{cases} 1 & \text{if } i = j; \\ -1 & \text{if } i = j \pm 1; \\ 1 & \text{if } i = j \pm 2; \\ 0 & \text{otherwise.} \end{cases}$

Problem 12 (*) Let $M_n$ be the number of perfect matching $m$ in the complete graph $K_{2n}$ such that

- $m$ contains no pair of crossing edges $(i, k), (j, l)$, for $i < j < k < l$.
- $m$ contains no pair of edges $(i, j), (i - 1, j + 1)$, for $i < j$.

Also let $\tilde{M}_n$ be the number of perfect matching $\tilde{m}$ in the complete graph $K_{2n}$ such that

- $\tilde{m}$ contains no pair of nesting edges $(i, l), (j, k)$, for $i < j < k < l$.
- $\tilde{m}$ contains no pair of edges $(i, j), (i + 1, j + 1)$, for $i < j$.

Show that $M_n = \tilde{M}_n$.

Problem 13 (*) (Kummer’s theorem) Let $m_n$ be the maximal power of 2 that divides the Catalan number $C_n = \frac{1}{n+1} \binom{2n}{n}$. Then $m_n + 1$ equals the sum of digits in the binary expansion of $n + 1$.

For example, $C_6 = 132$ is divisible by $2^2$ but not divisible by $2^3$ because $6 + 1 = 111$ (binary) has 3 digits 1.