Directions. You can collaborate, but must write up the solutions independently, i.e.,
thinking through the problem by yourself and expressing it in your own words. Consulting
solutions to problem sets of previous years or internet solutions is not allowed. Cite
significant theorems used.

Reading: 11.2-5. Limits of functions; Limit theorems; Limit form of continuity;
Discontinuities; Sequential continuity.

Problem 1. (2) Work 11.3/3a.
(One of the two inequalities needed to apply the Squeeze Theorem is a bit harder to find
than the other. Keep in mind what the variable is in the theorem.)

Problem 2. (1) Work 11.3/6ab. (Use limit theorems.)
Write the polynomial \( f(x) \) in the factored form:
\[ a_0 x^n \left( 1 + a_1 \left( \frac{1}{x} \right) + \ldots + a_n \left( \frac{1}{x} \right)^n \right). \]

Problem 3. (1) Prove (20) in Theorem 11.3C, the Limit Location Theorem for functions.
Try to do this without looking back at the proof of the corresponding theorem for sequences.

Problem 4. (2) Work 11.4/2, changing \([a, b]\) in both of its occurrences to \([a, b)\).
This shortens the work without sacrificing any of the ideas. Make two steps:
a) First just prove \( f(x) \) is increasing on \([a, b)\). Give a direct argument, using a limit
theorem; no indirect proofs. Begin by stating explicitly the minimal amount that you have
to prove.
b) Using part (a), prove \( f(x) \) is strictly increasing. The use of a “buffer” helps (like the
\( M \) in the proof of Theorem 7.4A).

The next two problems are exercises in the use of the Sequential Continuity Theorem
11.5: in working them, don’t go back to the basic \( \varepsilon - \delta \) definition of continuity.

Problem 5. (1)
Prove that if two functions \( f(x) \) and \( g(x) \) are continuous on \( \mathbb{R} \) and agree on all rational
points, i.e., \( f(a) = g(a) \) whenever \( a \) is a rational number, then \( f(x) = g(x) \) for all \( x \in \mathbb{R} \).
(You can cite and use results proved in Ass’t 2.)

Problem 6. (2) Work P11-2, assuming \( c > 0, c \neq 1 \).
(Hint: if it is a constant function, what must the constant value be? Start with an arbitrary
\( x \), and show \( f(x) \) has that value.)

((continued \( \rightarrow \))
Reading (Wed.): 12.1-12.2 Bolzano’s theorem; Intermediate value theorem. Locating and counting zeros.

Problem 7. (2) Let \( f(x) = x^3 + hx^2 - 1 \), where \( h > 0 \) and small. This polynomial has one zero \( z = z(h) \); its value depends on \( h \). Find its approximate value, and show by using Bolzano’s theorem and limit theorems that \( \lim_{h \to 0} z(h) \) exists.

(Follow the similar example in the book.)

Problem 8. (2: 1.5, .5)

a) Prove that the function \( x - \tan x \) has an infinity of positive zeros \( x_1, x_2, \ldots, x_n, \ldots \).

(Use the Intersection principle; you can use \( \infty \) informally in your argument and assume that the principle applies to it. Include a sketch of the relevant graphs.)

b) If \( N \) is large, approximately how many zeros are there \( \leq N \)?

Problem 9. (2) For what positive value of the parameter \( A \) will \( f(x) = x - A \sin x \) have exactly 4 positive zeros, for \( n > 0 \)?

Express this value of \( A \) in terms of the \( x_n \) of Problem 8.

(No formal proof is required, but show your reasoning and calculations.)

Optional Problem. (1 virtual gold star)

Try Problem 12-7 in the book – the Danish Ham Sandwich problem.

Use a separate sheet of paper (with your name). Hand it in on a separate pile. If you want to think about it over the long weekend (no class Monday), you can put it under my door (E18-314) any time Tues. Oct. 14. (Math Dept is locked and inaccessible Sat - Mon).