Flow polytopes of signed graphs and the Kostant partition function

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joint work with Karola Mészáros (Cornell)
Example of a type A flow polytope ($CRY(n)$)

$CRY(n) := \{ (b_{ij}) \in \mathbb{R}^{n^2} \mid \text{doubly-stochastic matrix, } b_{ij} = 0, i - j \geq 2 \}$

Example:

<table>
<thead>
<tr>
<th></th>
<th>$b_{11}$</th>
<th>$b_{12}$</th>
<th>$b_{13}$</th>
<th>$b_{14}$</th>
<th></th>
<th>$b_{21}$</th>
<th>$b_{22}$</th>
<th>$b_{23}$</th>
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<td>.4</td>
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<td>.6</td>
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<td>0</td>
<td>.6</td>
<td>.3</td>
<td>.1</td>
<td>0</td>
<td>.4</td>
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</tbody>
</table>

- $CRY(n)$ is the Chan-Robbins-Yuen polytope
- has $2^{n-1}$ vertices and $\dim(CRY(n)) = \binom{n}{2}$

Data: $v_n = \binom{n}{2}! \cdot \text{vol}(CRY(n))$

<table>
<thead>
<tr>
<th>$n$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_n$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>10</td>
<td>140</td>
<td>5880</td>
</tr>
<tr>
<td>$\frac{v_n}{v_{n-1}}$</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>14</td>
<td>42</td>
<td></td>
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Theorem [Zeilberger 99]:

$\binom{n}{2}! \cdot \text{vol}(CRY(n)) = \text{Cat}(0)\text{Cat}(1)\text{Cat}(2) \cdots \text{Cat}(n - 2)$. 
Example of a Kostant partition function

\[ f_n := \# \left\{ \text{ways of writing } (1, 2, \ldots, n - 1, -\binom{n}{2}) \text{ as} \right. \]
\[ \left. \text{N-combination of } e_i - e_j \right\} \]

Example:

\[ n = 2 : \quad (1, -1) = 1(1, -1) \quad \quad f_2 = 1 \]
\[ n = 3 : \quad (1, 2, -3) = 1(1, -1, 0) + 3(0, 1, -1) \]
\[ \quad = 1(1, 0, -1) + 2(0, 1, -1) \quad \quad f_3 = 2 \]
\[ n = 4 : \quad (1, 2, 3, -6) = 1(1, -1, 0, 0) + 3(0, 1, -1, 0) \]
\[ \quad + 6(0, 0, 1, -1) \]
\[ \quad = \cdots \quad \quad f_4 = 10 \]

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<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
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<th>4</th>
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Outline

1. What are type $A$ flow polytopes?

2. What are type $D$ flow polytopes?

3. How do we calculate volumes of flow polytopes?

4. Connection between type $A$ flow polytopes and Kostant partition function?

5. Is there such a connection for type $D$ flow polytopes?
From $\mathcal{CRY}(n)$ to flow polytopes $\mathcal{F}_G(\mathbf{a})$

$\mathcal{CRY}(n) := \{(b_{ij}) \in \mathbb{R}^{n^2} \mid \text{doubly-stochastic matrix, } b_{ij} = 0, i - j \geq 2\}$

<table>
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<tr>
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<tr>
<td></td>
<td>e</td>
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<td>0</td>
<td>h</td>
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<td>0</td>
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$K_5$

1 2 3 4 5
From $CRY(n)$ to flow polytopes $\mathcal{F}_G(a)$

$CRY(n) := \left\{(b_{ij}) \in \mathbb{R}^{n^2} \mid \text{doubly-stochastic matrix}, \ b_{ij} = 0, i - j \geq 2\right\}$

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\bullet & e & f & g \\
0 & \bullet & h & i \\
0 & 0 & \bullet & j \\
\end{array}
\]

\[1 = a + b + c + d\]
From $\mathcal{CRY}(n)$ to flow polytopes $\mathcal{F}_G(a)$

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\bullet & e & f & g \\
0 & \bullet & h & i \\
0 & 0 & \bullet & j
\end{array}$

$1 = a + b + c + d$

$0 = e + f + g - a$

$K_5$

1 \rightarrow a \rightarrow 0 \rightarrow e \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5
From $\mathcal{CRY}(n)$ to flow polytopes $\mathcal{F}_G(\mathbf{a})$

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$$
\begin{array}{|c|c|c|c|}
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a & b & c & d \\
\hline
\bullet & e & f & g \\
\hline
0 & \bullet & h & i \\
\hline
0 & 0 & \bullet & j \\
\hline
\end{array}
$$

$1 = a + b + c + d$

$0 = e + f + g - a$

$0 = h + i - b - e$

$K_5$
From $CRY(n)$ to flow polytopes $F_G(a)$

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\[
\begin{array}{c|cccc}
 a & b & c & d \\
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\end{array}
\]

\[
\begin{align*}
1 &= a + b + c + d \\
0 &= e + f + g - a \\
0 &= h + i - b - e \\
0 &= j - c - f - h
\end{align*}
\]
From \( \mathcal{CRY}(n) \) to flow polytopes \( \mathcal{F}_G(\mathbf{a}) \)

\[
\mathcal{CRY}(n) := \left\{ (b_{ij}) \in \mathbb{R}^{n^2} \mid \text{doubly-stochastic matrix, } b_{ij} = 0, i - j \geq 2 \right\}
\]

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<td>(j)</td>
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1 = \(a + b + c + d\)
0 = \(e + f + g - a\)
0 = \(h + i - b - e\)
0 = \(j - c - f - h\)

1 = -(\(j + i + q + g + d\))
From \( \mathcal{CRY}(n) \) to flow polytopes \( \mathcal{F}_G(a) \)

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\mathcal{CRY}(n) := \{ (b_{ij}) \in \mathbb{R}^{n^2} \mid \text{doubly-stochastic matrix, } b_{ij} = 0, i - j \geq 2 \}
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\[
\begin{align*}
1 &= a + b + c + d \\
0 &= e + f + g - a \\
0 &= h + i - b - e \\
0 &= j - c - f - h \\
-1 &= -(j + i + q + g + d)
\end{align*}
\]

Correspondence between \( \mathcal{CRY}(n) \) and flows in \( K_{n+1} \) with netflow: 1 first vertex, \(-1\) last vertex, 0 other vertices.

Example: (other graphs and netflow)

\[
\mathcal{F}_G((3, 4, -7)) \quad 3 = a + b + c \\
4 = d - c \\
-7 = -a - b - d
\]
From $CRY(n)$ to flow polytopes $\mathcal{F}_G(a)$

$CRY(n) := \{(b_{ij}) \in \mathbb{R}^{n^2} \mid \text{doubly-stochastic matrix, } b_{ij} = 0, i - j \geq 2\}$

$$
\begin{array}{cccc}
 a & b & c & d \\
\bullet & e & f & g \\
0 & \bullet & h & i \\
0 & 0 & \bullet & j
\end{array}
\begin{aligned}
1 &= a + b + c + d \\
0 &= e + f + g - a \\
0 &= h + i - b - e \\
0 &= j - c - f - h
\end{aligned}
\begin{aligned}
-1 &= -(j + i + q + g + d)
\end{aligned}

Correspondence between $CRY(n)$ and flows in $K_{n+1}$ with netflow: 1 first vertex, $-1$ last vertex, 0 other vertices.

Example: (other graphs and netflow)

$\mathcal{F}_G((3, 4, -7))$

$$
\begin{aligned}
3 &= a + b + c \\
4 &= d - c \\
-7 &= -a - b - d
\end{aligned}
$$

For graph $G$, vertices $\{1, 2, \ldots, n\}$, $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, the flow polytope of $G$ is $\mathcal{F}_G(a)$ (Postnikov-Stanley 05, Baldoni-Vergne 08)

$$
\mathcal{F}_G(a) := \{\text{flows } b(\epsilon) \in \mathbb{R}_{\geq 0}, \epsilon \in E(G) \mid \text{netflow vertex } i = a_i\}$$
Outline

1. What are type $A$ flow polytopes? ✓

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Flow polytopes for signed graphs

edges \( (i < j) \) correspond to \( e_i - e_j \) (roots in \( A_{n-1}^+ \))

we also consider:

edges \( i \) and \( j \) correspond to \( e_i + e_j \) and \( 2e_i \) (roots in \( C_n^+, D_n^+ \))

Example: (signed graphs)

\[
G^\pm \\
\begin{array}{ccc}
1 & e_1 - e_3 & 2 \\
\end{array}
\begin{array}{ccc}
e_1 + e_2 & 2e_2 & e_2 - e_3 \\
3 & 1 & 3
\end{array}
\]

\[ a = (1, 3, -2) \]

\[ 1 = a + b + c \]

\[ 3 = b + 2d + e - c \]

\[ -2 = -a - e \]

\( i.e. \ (1, 3, -2) = a \cdot (e_1 - e_3) + b \cdot (e_1 + e_2) + c \cdot (e_1 - e_2) + \cdots \)

\( G^\pm \) graph with edges \( i \) \( j \) \( i \) and \( \{1, 2, \ldots, n\} \) vertices, \( a = (a_1, \ldots, a_n) \in \mathbb{Z}^n \), the signed flow polytope of \( G^\pm \) is

\[
\mathcal{F}_{G^\pm}(a) := \{ \text{flows } b(\epsilon) \in \mathbb{R}_{\geq 0}, \ \epsilon \in E(G^\pm) \mid \text{netflow vertex } i = a_i \}.
\]
Examples flow polytopes

\[ a = (1, -1) \]
\[ 1 = a + b + c + d \]
\[ a, b, c, d \geq 0 \]

simplex

\[ a = (2) \]
\[ 2 = 2a + 2b + 2c + 2d + 2e \]
\[ a, b, c, d, e \geq 0 \]

simplex

\[ K_4 \]
\[ 1 \]
\[ 0 \]
\[ 0 \]
\[ -1 \]

\[ K_4^\pm \]
\[ 2 \]
\[ 0 \]
\[ 0 \]
\[ 0 \]

\text{CRY}

\text{type D CRY}
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Volumes and triangulations

- \( \mathcal{P} \subset \mathbb{R}^n \) convex polytope, \( \dim(\mathcal{P}) = n \),

- A **triangulation** \( T \) is collection of \( n \)-simplices:
  (i) \( \mathcal{P} = \bigcup_{\Delta \in T} \Delta \),
  (ii) for \( \Delta, \Delta' \in T \), \( \Delta \cap \Delta' \) is face common to \( \Delta, \Delta' \).

Triangulation into 6 \( \Delta \)s.

- when \( T \) is indexed by **combinatorial objects**
  \( \Rightarrow \) normalized volume of \( \mathcal{P} = \#T = \# \text{ objects} \).
- we triangulate \( \mathcal{F}_{G^\pm} \), triangulation indexed by certain **integral flows**
on \( G^\pm \).
Triangulating flow polytopes

underlying relation:

\[(e_i - e_j) + (e_j - e_k) = e_i - e_k.\]
Triangulating flow polytopes

underlying relation:

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Triangulating flow polytopes

underlying relation:

$$(e_i - e_j) + (e_j - e_k) = e_i - e_k.$$
Triangulating flow polytopes

underlying relation:

$$(e_i - e_j) + (e_j - e_k) = e_i - e_k.$$  

underlying relation:

$$(e_i + e_j) + (e_k - e_j) = e_i + e_k.$$  

Proposition:  

$$\mathcal{F}_{G^\pm}(a) = \mathcal{F}_{G^\pm_1}(a) \cup \mathcal{F}_{G^\pm_2}(a).$$  

- $G^\pm_1$ and $G^\pm_2$ have one fewer edge incident to $j$.  
- iterating proposition on vertex with zero flow:
Proposition: $\mathcal{F}_{G^\pm}(a) = \mathcal{F}_{G^\pm_1}(a) \cup \mathcal{F}_{G^\pm_2}(a)$.

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**Triangulating flow polytopes**

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Triangulating flow polytopes

\[ G^\pm \]

underlying relation:
\[(e_i - e_j) + (e_j - e_k) = e_i - e_k.\]

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\[(e_i + e_j) + (e_k - e_j) = e_i + e_k.\]

Proposition: \[ \mathcal{F}_{G^\pm}(a) = \mathcal{F}_{G^\pm_1}(a) \cup \mathcal{F}_{G^\pm_2}(a). \]

- \(G^\pm_1\) and \(G^\pm_2\) have one fewer edge incident to \(j\).
- iterating proposition on vertex with zero flow:
Integral flows / lattice points: Kostant partition function

\[ \mathcal{F}_{G^\pm}(a) := \{ \text{flows } b(\epsilon) \in \mathbb{R}_{\geq 0}, \, \epsilon \in E(G^\pm) \mid \text{netflow vertex } i = a_i \} \]

Interpret \( E(G^\pm) \) as multiset of roots:

\begin{align*}
\{ \text{lattice points of } \mathcal{F}_{G^\pm}(a) \} & \equiv \# \{ \text{ways of expressing } a \text{ as an } \mathbb{N}\text{-combination of roots of } G^\pm \} \\
\text{integral flows netflow } a
\end{align*}

We call this number: \( K_{G^\pm}(a) \), the **Kostant partition function**.

**Example:**

\[ G^\pm \quad a = (1, 3, -2) \]

\[ K_{G^\pm}((1, 3, -2)) = 3, \quad \text{since:} \]

\[ (1, 3, -2) = 1(e_1 - e_3) + 1(2e_2) + 1(e_2 - e_3) \]

\[ = 1(e_1 + e_2) + 2(e_2 - e_3) \]

\[ = 1(e_1 - e_2) + 1(2e_2) + 2(e_2 - e_3) \]
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5. Is there such a connection for type $D$ flow polytopes?
Volume of $\mathcal{F}_G(1, 0, \ldots, 0, -1)$

**Theorem [Postnikov-Stanley 00]:**
For a graph $G$, vertices $\{1, 2 \ldots, n\}$, only negative edges

$$\dim(\mathcal{F}_G)! \cdot \text{vol} (\mathcal{F}_G(1, 0, \ldots, 0, -1)) = K_G(0, d_2, \ldots, d_{n-1}, -\sum_{i=2}^{n-1} d_i),$$

where $d_i = (\text{indegree of } i) - 1.$
Theorem [Postnikov-Stanley 00]:
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where $d_i = (\text{indegree of } i) - 1$.

Example:

Volume of flow polytope $\mathcal{F}_G(1, 0, 0, -1)$ for

$$= K_G(0, 3, 2, -5),$$
$$= \#\text{integral flows in}$$

Note:
$\text{vol}(\mathcal{F}_G(1, 0, \ldots, 0, -1))$ given by $\#\text{ lattice points of } \mathcal{F}_G(0, d_2, d_3, \ldots)$.
Application to $\mathcal{CRY}(n)$

Since $\mathcal{CRY}(n) = \mathcal{F}_{K_{n+1}}(1, 0, \ldots, 0, -1)$, then

**Corollary**

$${n \choose 2}! \cdot \text{vol}(\mathcal{CRY}(n)) = K_{K_{n+1}}(0, 0, 1, 2, \ldots, n - 2, -\left(\frac{n-1}{2}\right))$$

$$= K_{K_{n-1}}(1, 2, \ldots, n - 2, -\left(\frac{n-1}{2}\right))$$ (†)

**Example:**

$$6! \cdot \text{vol}(\mathcal{CRY}(4)) = K_{K_3}(1, 2, -3) = 2:$$

$$(1, 2, -3) = 1(e_1 - e_2) + 3(e_2 - e_3) = 1(e_1 - e_3) + 2(e_2 - e_3)$$
Application to \( \mathcal{CRY}(n) \)

Since \( \mathcal{CRY}(n) = \mathcal{F}_{K_{n+1}}(1, 0, \ldots, 0, -1) \), then

\[ \binom{n}{2}! \cdot \text{vol}(\mathcal{CRY}(n)) = K_{K_{n+1}}(0, 0, 1, 2, \ldots, n-2, -\binom{n-1}{2}) \]
\[ = K_{K_{n-1}}(1, 2, \ldots, n-2, -\binom{n-1}{2}) \] (†)

**Example:**

\[ 6! \cdot \text{vol}(\mathcal{CRY}(4)) = K_{K_3}(1, 2, -3) = 2: \]
\[ (1, 2, -3) = 1(e_1 - e_2) + 3(e_2 - e_3) = 1(e_1 - e_3) + 2(e_2 - e_3) \]

**Remarks:**

- Zeilberger used (†), the generating series of \( K_G(a) \), and the *Morris Identity* to calculate \( \binom{n}{2}! \cdot \text{vol}(\mathcal{CRY}(n)) = \prod_{i=0}^{n-2} \text{Cat}(i) \),
- No combinatorial proof for this formula of \( \text{vol}(\mathcal{CRY}(n)) \).
Idea proof of Theorem on $\text{vol} \mathcal{F}_G(e_1 - e_n)$

$$\text{vol}(\mathcal{F}_G(a)) = \frac{1}{\dim(\mathcal{F}_G)!} \# \{ \text{ } \}$$

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$$\text{vol}(\mathcal{F}_G(a)) = \frac{1}{\dim(\mathcal{F}_G)!} \# \{ \text{ } \}$$
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Dynamic Flow

For signed graphs: \( \text{vol}(\mathcal{F}_{G^\pm}(2e_1)) \neq \#\{\text{integral flows on } G^\pm\} = K_{G^\pm}(\cdot) \)

\[ = \#\{\text{integral dynamic flows on } G^\pm\} \]
Dynamic Flow

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- split edges into two half-edges
Dynamic Flow

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- split edges \((i, j)\) into two half-edges
- if left half-edge has flow \(k\)
  \(\rightarrow\) add \(k\) new right half-edges

\[ k \rightarrow \begin{cases} \cdot \\ \end{cases} \]
Dynamic Flow

For signed graphs: \( \text{vol}(\mathcal{F}_{G^\pm}(2e_1)) \neq \#\{\text{integral flows on } G^\pm\} = K_{G^\pm}(\cdot) = \#\{\text{integral dynamic flows on } G^\pm\} \)

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Example:
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Example:

\[G^\pm\]
Dynamic Flow

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Example:

\( G^\pm \)

\[
\begin{array}{c}
\text{2} & \text{1} & \text{1} \\
\text{2} & \text{1} & \text{1} \\
\end{array}
\]
Dynamic Flow

For signed graphs: $\text{vol}(\mathcal{F}_{G^\pm}(2e_1)) \neq \#\{\text{integral flows on } G^\pm\} = K_{G^\pm}(\cdot)$

$= \#\{\text{integral dynamic flows on } G^\pm\}$

- split edges $i \rightarrow j$ into two half-edges
- if left half-edge has flow $k$
  $\rightarrow$ add $k$ new right half-edges

Example:

we define: $K^{\text{dyn.}}_{G^\pm}(a) := \#\{\text{integral dynamic flows in } G^\pm, \text{ netflow } a\}$
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Generating function:

$$\sum_{a \in \mathbb{Z}^n} K_{G^\pm}(a)x^a = \prod_{i, j \in E(G^\pm)} \left(1 - x_i x_j^{-1}\right)^{-1} \prod_{i, j \in E(G^\pm)} \left(1 - x_i x_j\right)^{-1}. $$
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\[
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G^\pm & \\
2 & \rightarrow & 1 & \rightarrow & 1 & \rightarrow & 1 & \rightarrow & 1
\end{align*}
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Volume of $\mathcal{F}_{G^\pm}(2, 0, \ldots, 0)$

**Theorem [Mészáros-M 11]:**
For a signed graph $G^\pm$, vertices $\{1, 2 \ldots, n\}$

\[
\dim(\mathcal{F}_{G^\pm})! \cdot \text{vol}(\mathcal{F}_{G^\pm}(2, 0, \ldots, 0)) = K_{G^\pm}^{\text{dyn.}}(0, d_2, \ldots, d_{n-1}, d_n),
\]

where $d_i = \text{(indegree of } i\text{)} - 1$. 
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\[0 \quad 1 \quad 0 \quad 1\]
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$= 5$:
Idea proof of Theorem on $\text{vol}\mathcal{F}_G^\pm(2e_1)$

$$\text{vol}(\mathcal{F}_G(a)) = \frac{1}{\dim(\mathcal{F}_G)!} \# \left\{ \begin{array}{c} \includegraphics[width=.7\textwidth]{flower.png} \end{array} \right\}$$
Application to type D analogue of $CRY(n)$

Recall $CRY(n) = \mathcal{F}_{K_{n+1}}(e_1 - e_{n+1})$:

- dimension $\binom{n}{2}$, $2^{n-1}$ vertices, volume $\prod_{i=0}^{n-2} \text{Cat}(i)$. 
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$$(n(n-2))! \cdot \text{vol}(\mathcal{CRY}^\pm(n)) = K_{K_n^\pm}^{\text{dyn.}}(0, 0, 1, 2, \ldots, n-3, n-2).$$
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**Data:** $v_n = \dim(\mathcal{CRY}^\pm(n))! \cdot \text{vol}(\mathcal{CRY}^\pm(n))$

<table>
<thead>
<tr>
<th>$n$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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</tr>
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<td>1</td>
<td>2</td>
<td>32</td>
<td>5120</td>
<td>9175040</td>
<td>197300060160</td>
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**Conjecture** $v_n = 2^{(n-2)^2} \cdot \text{Cat}(0)\text{Cat}(1)\text{Cat}(2) \cdots \text{Cat}(n-2)$. 
Outline

1. What are type $A$ flow polytopes? ✓

2. What are type $D$ flow polytopes? ✓

3. How do we calculate volumes of flow polytopes? ✓

4. Connection between type $A$ flow polytopes and Kostant partition function? ✓

5. Is there such a connection for type $D$ flow polytopes? ✓
References:

- C. De Concini, C. Procesi, **Topics in Hyperplane Arrangements, Polytopes and Box Splines**, Springer 2011
- with K. Mészáros, **Flow polytopes of signed graphs and the Kostant partition function**, arXiv:1208.0140, code at sites.google.com/site/flowpolytopes/

THANK YOU