Bijectons and symmetries for factorizations of a long cycle and Jackson’s formula

Alejandro Morales (MIT)

Séminaire de Combinatoire du LIAFA

February 16, 2012

joint work with Olivier Bernardi (MIT, CNRS)
Teaser: a symmetry of plane trees

Plane trees on $n + 1$ vertices are counted by the Catalan numbers $C_n = \frac{1}{n+1} \binom{2n}{n}$

Example:

- $n = 3$

  - $\mathcal{T}_{2 \lambda}^{2,1} = \mathcal{T}_{2 \lambda}^{2,1} = \mathcal{T}_{3,1}^{1,3}$

  - $\mathcal{T}_{2 \lambda}^{2,1} = \mathcal{T}_{2 \lambda}^{2,1} = \mathcal{T}_{3,1}^{1,3}$

- $n = 4$

  - $\mathcal{T}_{2 \lambda}^{2,1} = \mathcal{T}_{2 \lambda}^{2,1} = \mathcal{T}_{3,1}^{1,3}$

  - $\mathcal{T}_{2 \lambda}^{2,1} = \mathcal{T}_{2 \lambda}^{2,1} = \mathcal{T}_{3,1}^{1,3}$

- $\mathcal{T}_\lambda$: set of plane trees vertex degree $\lambda \vdash 2n$, $\ell(\lambda) = n + 1$.
- If $\lambda = [1^{m_1}, 2^{m_2}, \ldots]$, let $\text{Aut}(\lambda) = \prod_i m_i!$.

Proposition: $\text{Aut}(\lambda) \cdot \# \mathcal{T}_\lambda = \text{Aut}(\mu) \cdot \# \mathcal{T}_\mu = n! \cdot 2$

- this is an instance of symmetry of colored factorizations.
Ordered factorizations of the cycle \((1 2 \ldots n)\)

- \(S_n\) be the group of permutations \(\pi\) of \([n] = \{1, 2, \ldots, n\}\).
- \(c(\pi) = \#\{\text{cycles of } \pi\}\), and \(\gamma = (1 2 \ldots n)\).

\[
K_{q_1,\ldots,q_k}^n := \{(\pi_1, \ldots, \pi_k) \mid \pi_1 \circ \cdots \circ \pi_k = \gamma, c(\pi_i) = q_i\},
\]
\[
K_{q_1,\ldots,q_n}^n := \#K_{q_1,\ldots,q_k}^n
\]

Examples

\(k = 3, n = 2\), if \(\iota = (1)(2), \quad K_{1,1,1}^2 = 1,\ K_{1,2,2}^2 = K_{2,1,2}^2 = K_{2,2,1}^2 = 1\).

\[
(1 2) = (1 2) \circ (1 2) \circ (1 2) = (1 2) \circ \iota \circ \iota = \iota \circ (1 2) \circ \iota = \iota \circ \iota \circ (1 2).
\]

\(k = 3, n = 3, q_1 = 2, q_2 = 1, q_3 = 2\).

\[
(1 2 3) = (1 2)(3) \circ (1 3 2) \circ (1 2)(3) = (1 3)(2) \circ (1 3 2) \circ (1 3)(2)
\]
\[
= (2 3)(1) \circ (1 3 2) \circ (2 3)(1) = (1 2)(3) \circ (1 2 3) \circ (1 3)(2)
\]
\[
= (1 3)(2) \circ (1 2 3) \circ (2 3)(1) = (2 3)(1) \circ (1 2 3) \circ (1 2)(3).
\]
Colored factorizations of $\gamma = (1 \ 2 \ \ldots \ n)$

$K_{q_1,\ldots,q_k}^n$ are hard to compute directly (complicated sums of characters). But generating series has a nice interpretation: evaluate $x_i = n_i \in \mathbb{N}$

$$\sum_{q_i \geq 0} K_{q_1,\ldots,q_k}^n n_1^{q_1} \cdots n_k^{q_k} = \sum_{p_i \geq 0} C_{p_1,\ldots,p_k}^n \binom{n_1}{p_1} \cdots \binom{n_k}{p_k}.$$ 

$C_{p_1,\ldots,p_k}^n := \# \mathcal{P}^{[p_i]}(\pi_1 \cdots \pi_k = \gamma, \text{cycles})$, where $\mathcal{P}^{[p_i]}(\pi_i)$ is the number of factorizations of $\pi_i$ colored with all colors of $[p_i]$.

Example

$k = 3, n = 2$:

$C_{1,1,1}^2 = 4$

$(1\ 2) \circ (1\ 2) \circ (1\ 2)$

$(1\ 2) \circ (1)(2) \circ (1)(2)$

$(1)(2) \circ (1\ 2) \circ (1)(2)$

$(1)(2) \circ (1\ 2) \circ (1\ 2)$

$C_{2,1,1}^2 = 4$

$C_{2,2,2}^2 = 0$
Jackson’s formula for colored factorizations

\[ C_{p_1, \ldots, p_k}^n := \left\{ \text{factorizations } \pi_1 \cdots \pi_k = \gamma, \text{ cycles of } \pi_i \text{ colored with all colors of } [p_i] \right\}, \quad C_{p_1, \ldots, p_k}^m := \#C_{p_1, \ldots, p_k}^n. \]

\[ M_{r_1, \ldots, r_k}^m := \left\{ (S_1, \ldots, S_n) \mid S_i \subsetneq [k], \ r_j \text{ sets } S_i \text{ contain } j \right\}, \quad M_{r_1, \ldots, r_k}^m := \#M_{r_1, \ldots, r_k}^m. \]

Theorem [Jackson 88]

\[ C_{p_1, \ldots, p_k}^m = n!^{k-1} \cdot M_{p_1-1, \ldots, p_k-1}^{n-1}. \]

Remarks

- Proof is algebraic, uses irreducible characters of the symmetric group:

\[ K_{q_1, \ldots, q_k}^n = n!^{k-1} \sum_{\lambda_i \vdash n, \ell(\lambda_i) = q_i} \sum_r (-1)^r (r!(n-1-r)!)^{k-1} \prod_{i=1}^k \chi_{n-r, 1^r}^{\lambda_i}. \]
Combinatorial proofs Jackson’s formula $k = 2, 3$

**Theorem [Jackson 88]**

\[
\# \left\{ \pi_1 \cdots \pi_k = \gamma, \text{ cycles of } \pi_i \right\} = n^{k-1} \cdot \# \left\{ (S_1, \ldots, S_{n-1}), \ S_i \not\subset [k], \ p_j - 1 \text{ sets } S_i \text{ contain } j \right\}
\]

\[
C^n_{p_1, \ldots, p_k} \downarrow \quad M^{n-1}_{p_1-1, \ldots, p_k-1}
\]

Bijective computation of $\#C^n_{p_1, \ldots, p_k}$ when $k = 2, 3$:

\[
C^n_{p_1, p_2} = n! \cdot \binom{n-1}{p_1-1, p_2-1}
\]

\[
= n! \cdot \# \left\{ (S_1, \ldots, S_{n-1}) \mid S_i \not\subset \{1, 2\}, p_j - 1 \text{ sets } S_i \text{ contain } j \right\}
\]

\[
C^n_{p_1, p_2, p_3} = n!^2 \cdot \binom{n-1}{p_3-1} \sum_{a \geq 0} \binom{n-p_2}{p_1-a-1} \binom{n-p_3}{a} \binom{n-1-a}{n-p_2}.
\]

\[
= n!^2 \cdot \# \left\{ (S_1, \ldots, S_{n-1}) \mid S_i \not\subset \{1, 2, 3\}, p_j - 1 \text{ sets } S_i \text{ contain } j \right\}
\]

We prove this result for all $k$ combinatorially.
Outline of talk

I Background on Jackson’s formula ✓

II Symmetry of colored factorizations

III Probabilistic puzzle equivalent to Jackson’s formula

IV Bijectons for symmetry and towards probabilistic puzzle
Colored factorizations by type

\[ C_{p_1, \ldots, p_k}^n := \left\{ \text{factorizations } \pi_1 \cdots \pi_k = \gamma, \text{ cycles} \right\}, \quad C_{p_1, \ldots, p_k} := \#C_{p_1, \ldots, p_k}. \]

Let \( \alpha^{(i)} = (\alpha^{(i)}_1, \alpha^{(i)}_2, \ldots, \alpha^{(i)}_{p_i}) \) be a composition of \( n \)

\[ C_{\alpha^{(1)}, \ldots, \alpha^{(k)}} := \left\{ \text{factorizations } \pi_1 \cdots \pi_k = \gamma, \right\}
\[ \left\{ \text{cycles of } \pi_i \text{ are colored, } \pi_i \text{ has } \alpha^{(i)}_j \text{ elements colored } j \right\}, \quad C_{\alpha^{(1)}, \ldots, \alpha^{(k)}} := \#C_{\alpha^{(1)}, \ldots, \alpha^{(k)}}. \]

**Example** \( k = 2, n = 4: \)

\( (1, 3)(2)(4)(5) \circ (1, 4)(2, 5, 3) \circ (1, 3)(2, 4)(5) \in C_{(2,1,2),(5),(3,2)} \)

- \( C_{p_1, \ldots, p_k}^n = \sum_{\alpha^{(i)} | = n, \ell(\alpha^{(i)}) = p_i} C_{\alpha^{(1)}, \ldots, \alpha^{(k)}}. \)

- Refinements of bijections of Schaeffer-Vassilieva 07,08 give \( C_{\alpha^{(1)}, \alpha^{(2)}} \) and \( C_{\alpha^{(1)}, \alpha^{(2)}, \alpha^{(3)}}. \) (M-Vassilieva 09)
Symmetry of colored factorizations by type

\[ C_{\alpha(1), \ldots, \alpha(k)} := \left\{ \begin{array}{l} \text{factorizations } \pi_1 \cdots \pi_k = \gamma, \\ \text{cycles of } \pi_i \text{ are colored, } \pi_i \text{ has } \alpha_j^{(i)} \text{ elements colored } j \end{array} \right\}, \quad C_{\alpha(1), \ldots, \alpha(k)} := \#C_{\alpha(1), \ldots, \alpha(k)} \]

Formulas for \( k = 2, 3 \) suggest \( C_{\alpha(1), \ldots, \alpha(k)} \) only depends on \( n \) and \( \ell(\alpha(i)) \).

**Example** \( k = 2, n = 4 \):

<table>
<thead>
<tr>
<th>( \alpha^{(1)} )</th>
<th>( \alpha^{(2)} )</th>
<th>4</th>
<th>31</th>
<th>22</th>
<th>211</th>
<th>1111</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>24</td>
<td>24</td>
<td>24</td>
<td>24</td>
<td>24</td>
<td>24</td>
</tr>
<tr>
<td>31</td>
<td>24</td>
<td>16</td>
<td>16</td>
<td>8</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>22</td>
<td>24</td>
<td>16</td>
<td>16</td>
<td>8</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>211</td>
<td>24</td>
<td>8</td>
<td>8</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>11111</td>
<td>24</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

**Theorem** [M-Vassilieva \( k = 2, 3 \), Bernardi-M all \( k \)]

\( \alpha^{(1)}, \ldots, \alpha^{(k)} \) and \( \beta^{(1)}, \ldots, \beta^{(k)} \) compositions of \( n \) with 
\( \ell(\alpha^{(i)}) = \ell(\beta^{(i)}) = p_i \) then \( C_{\alpha^{(1)}, \ldots, \alpha^{(k)}} = C_{\beta^{(1)}, \ldots, \beta^{(k)}} \).

**Corollary:** 

\[ C_{\alpha^{(1)}, \ldots, \alpha^{(k)}} = \frac{C_{p_1, \ldots, p_k}^n}{\prod_{i=1}^k \binom{n-1}{p_i-1}} = \frac{n!^{k-1} \cdot M_{p_1-1, \ldots, p_k-1}^{n-1}}{\prod_{i=1}^k \binom{n-1}{p_i-1}} \]
Outline of talk

I Background on Jackson’s formula ✓

II Symmetry of colored factorizations ✓

III Probabilistic puzzle equivalent to Jackson’s formula

IV Bijectons for symmetry and towards probabilistic puzzle

V Solving probabilistic puzzle
Probabilistic puzzle equivalent to Jackson’s formula

Chain graph on $\mathcal{M}_{p_1,\ldots,p_k}$

- For $j \in [k]$ and $S \subseteq [k]$ define
  \[ n(j, S) = \begin{cases} 
  j - 1 \pmod{k} & \text{if } j \in S \\
  j & \text{if } j, j + 1 \notin S \\
  j + r \pmod{k} & \text{if } j, j + r + 1 \notin S \\
  j + 1, \ldots, j + r \in S. 
  \end{cases} \]

- For $S = (S_1, \ldots, S_n) \in \mathcal{M}_{p_1,\ldots,p_k}$ and $A = (a_1, \ldots, a_{k-1}) \in [n]^{k-1}$, let $G(A, S)$ be graph:
  - vertices $[k]$ and edges $\{e_1, \ldots, e_{k-1}\}$, $e_j = \{j, n(j, S_{a_j})\}$.

Example ($n=6$, $k=4$)

$S = \begin{array}{ccccccc}
\text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } \\
\end{array}$

$A = (3, 3, 5)$

$G(A, S) =$

$n(1, S_3) = 1 + 3, \ n(2, S_3) = 2 - 1, \ n(3, S_5) = 3,$

edges: $\{1, n(1, S_3)\} = \{1, 4\}, \ \{2, n(2, S_3)\} = \{2, 1\}, \ \{3, n(3, S_1)\} = \{3, 3\}.$
Probabilistic puzzle equivalent to Jackson’s formula

Chain graph on $\mathcal{M}_p^n$ for $p_1, \ldots, p_k$

- For $j \in [k]$ and $S \subseteq [k]$ define
  
  \[ n(j, S) = \begin{cases} 
  j - 1 \pmod{k} & \text{if } j \in S \\
  j & \text{if } j, j + 1 \notin S \\
  j + r \pmod{k} & \text{if } j, j + r + 1 \notin S \\
  j + 1, \ldots, j + r \in S.
  \end{cases} \]

- For $S = (S_1, \ldots, S_n) \in \mathcal{M}_p^n$ and $A = (a_1, \ldots, a_{k-1}) \in [n]^{k-1}$, let $G(A, S)$ be graph:
  - vertices $[k]$ and edges $\{e_1, \ldots, e_{k-1}\}$, $e_j = \{j, n(j, S_{a_j})\}$.

Example ($n=6$, $k=4$)

$S = \begin{array}{cccc}
1 & 1 & 1 \\
\times & \times & \times \\
\times & \times & \times \\
\end{array}$

$A' = (3, 3, 2)$

$G(A', S) = \begin{array}{ccc}
1 & 4 \\
\times & \times \\
2 & \times \\
\end{array}$

$n(1, S_3) = 1 + 3, \quad n(2, S_3) = 2 - 1, \quad n(3, S_2) = 3 + 3 \equiv 2 \pmod{4}$

edges: $\{1, n(1, S_3)\} = \{1, 4\}, \quad \{2, n(2, S_3)\} = \{2, 1\}, \quad \{3, n(3, S_1)\} = \{3, 2\}$. 
Probabilistic puzzle equivalent to Jackson’s formula

Chain graph on $\mathcal{M}^n_{p_1,\ldots,p_k}$ (fix $k; n, p_1, \ldots, p_k$)

- For $j \in [k]$ and $S \subseteq [k]$ define $n : [k] \times \mathcal{M}^n_{r_1,\ldots,r_k}$ as before.

- For $S = (S_1, \ldots, S_n) \in \mathcal{M}^n_{p_1,\ldots,p_k}$ and $A = (a_1, \ldots, a_{k-1}) \in [n]^{k-1}$, let $G(A, S)$ be graph: vertices $[k]$, edges $e_j = \{j, n(j, S_{a_j})\}$ $j = 1$ to $k-1$.

Examples

$A = (3, 3, 5) \quad A' = (3, 3, 2)$

$G(A, S) = \begin{array}{c}
1 \quad 2 \quad 3 \quad 4 \\
\end{array}$

$G(A', S) = \begin{array}{c}
1 \quad 2 \quad 3 \quad 4 \\
\end{array}$

Theorem (probabilistic puzzle) [Bernardi-M 11]
Consider uniform distribution on $(A, S)$, for $A = (a_1, \ldots, a_{k-1})$ in $[n]^{k-1}$ and $S$ in $\mathcal{M}^n_{p_1,\ldots,p_k}$. Then

$$\Pr(G(A, S) \text{ is a tree}) = \Pr(\#S_1 = k - 1).$$

Theorem [Bernardi-M 11]
Probabilistic puzzle is bijectively equivalent to Jackson’s theorem.
Probabilistic puzzle for \(k = 2\)

**Theorem (probabilistic puzzle) [Bernardi-M 11]**
Consider uniform distribution on \((A, S)\), for \(A = (a_1, \ldots, a_{k-1})\) in \([n]^{k-1}\) and \(S\) in \(M^n_{p_1, \ldots, p_k}\). Then
\[
\Pr(G(A, S) \text{ is a tree}) = \Pr(\#S_1 = k - 1).
\]

**Proof (k=2):**

Need \(G = T = \begin{array}{c}
1 \\
2
\end{array}\).

\[
S = \begin{array}{cccccc}
\square & \square & \square & \square & \square & \square
\end{array}
\]

\(A = a\) in \([n]\).

need \(e_1 = \{1, n(1, S_a)\} = \{1, 2\}\),

\(n(1, S_a) = 2\) iff \(S_a = \{1\} = \begin{array}{c}
\square
\end{array}\) or \(S_a = \{2\} = \begin{array}{c}
\square
\end{array}\)

Then \(\Pr(G(A, S) \text{ is a tree}) = \Pr(\#S_a = 1) = \Pr(\#S_1 = 2 - 1).\)
Probabilistic puzzle for $k = 3$

Theorem (probabilistic puzzle) [Bernardi-M 11]
Consider uniform distribution on $(A, S)$, for $A = (a_1, \ldots, a_{k-1})$ in $[n]^{k-1}$ and $S$ in $\mathcal{M}_n^{p_1, \ldots, p_k}$. Then

$$\Pr(G(A, S) \text{ is a tree}) = \Pr(\#S_1 = k - 1).$$

Proof ($k = 3$):

Need $G =$. 

```
  1 -- 2 -- 3
```

or

```
  1 -- 2 -- 3
```

or

```
  1 -- 2
```

$S = [\text{matrix}]$ 

$A = (a, b) \text{ in } [n] \times [n]$.

If $P_{R \times R'} = \Pr(R \subseteq S_i \text{ and } R' \subseteq S_j)$ then

$$\Pr(G(A, S) \text{ is a tree}) = P_{\{1\} \times \{2\}} + P_{\{1\} \times \{3\}} + P_{\{2\} \times \{3\}} - P_{\{1\} \times \{2,3\}} - P_{\{2\} \times \{1,3\}} - P_{\{3\} \times \{1,2\}},$$

$$+ P_{\{1,2\} \times \{1,3\}} + P_{\{1,2\} \times \{2,3\}} + P_{\{1,3\} \times \{2,3\}}.$$ 

(by inclusion-exclusion)
Probabilistic puzzle for $k = 3$

Proof ($k = 3$) continued:
If $P_{R \times R'} = \Pr(R \subseteq S_i \text{ and } R' \subseteq S_j)$ then
\[
\Pr(G(A, S) \text{ is a tree}) = P_{\{1\} \times \{2\}} + P_{\{1\} \times \{3\}} + P_{\{2\} \times \{3\}} - P_{\{1\} \times \{2,3\}} - P_{\{2\} \times \{1,3\}} - P_{\{3\} \times \{1,2\}},
\]
\[
+P_{\{1,2\} \times \{1,3\}} + P_{\{1,2\} \times \{2,3\}} + P_{\{1,3\} \times \{2,3\}}.
\]

Exchange Lemma ($k = 3$): For $\{a, b, c\} = \{1, 2, 3\}$,
\[
P_{\{a,b\} \times \emptyset} = P_{\{a\} \times \{b\}} - P_{\{a,c\} \times \{b\}} + P_{\{a,b\} \times \{a,c\}}.
\]

Why? $P_{\{a,b\} \times \emptyset} - P_{\{a,b\} \times \{a,c\}} = P_{\{a\} \times \{b\}} - P_{\{a,c\} \times \{b\}}$.

Then
\[
\Pr(G(A, S) \text{ is a tree}) = P_{\{1,2\} \times \emptyset} + P_{\{1,3\} \times \emptyset} + P_{\{2,3\} \times \emptyset} = \Pr(#S_1 = 2).
\]
On proof of probabilistic puzzle for all $k$

- For $k = 4$ there are 16 possible trees... find more Exchange Lemmas... use Gröbner bases... Instead
  - For all $k$, by matrix-tree theorem get $\Pr(G \text{ is tree})$ as a sort of determinant,
  - Use multi-linear algebra and bijections to show $\Pr(G \text{ is tree})$ is equivalent to

$$P_{[k]\{1\}} \times \emptyset \cdots + \cdots + P_{[k]\{k\}} \times \emptyset \cdots = \Pr(\#S_1 = k - 1).$$
Outline of talk

I Background on Jackson’s formula ✓

II Symmetry of colored factorizations ✓

III Probabilistic puzzle equivalent to Jackson’s formula ✓

IV Bijectons for symmetry and towards probabilistic puzzle

0. From colored factorizations to rooted colored unicellular constellations,
How to represent factorizations

For $k = 3$, $[n]$:

cycles $\pi_1$:  

cycles $\pi_2$:  

cycles $\pi_3$:  

How to represent factorizations

For $k = 3$, $[n]$:

- $\pi_1 = (1)(2\ 3)(4)(5)$
- $\pi_2 = (1\ 2)(3)(4)(5)$
- $\pi_3 = (1\ 3)(2\ 5\ 4)$

$\pi_1 \pi_2 \pi_3 = (1\ 2\ 5\ 4)(3)$. 
How to represent factorizations

For $k = 3$, $[n]$:

- **cycles $\pi_1$:**
- **cycles $\pi_2$:**
- **cycles $\pi_3$:**

Examples

\[
\begin{align*}
\pi_1 &= (1)(2\,3)(4)(5) \\
\pi_2 &= (1\,2)(3)(4)(5) \\
\pi_3 &= (1\,3)(2\,5\,4) \\
\pi_1\pi_2\pi_3 &= (1\,2\,5\,4)(3).
\end{align*}
\]
How to represent factorizations

For $k = 3$, $[n]$:

- cycles $\pi_1$: 
  - $\pi_1 = (1)(2\ 3)(4)(5)$
- cycles $\pi_2$: 
  - $\pi_2 = (1\ 2)(3)(4)(5)$
- cycles $\pi_3$: 
  - $\pi_3 = (1\ 3)(2\ 5\ 4)$

Examples:

$\pi_1\pi_2\pi_3 = (1\ 2\ 5\ 4)(3)$. 
How to represent factorizations

For $k = 3$, $[n]:

cycles $\pi_1:$

cycles $\pi_2:$

cycles $\pi_3:$

Examples

$\pi_1 = (1)(2\ 3)(4)(5)$

$\pi_2 = (1\ 2)(3)(4)(5)$

$\pi_3 = (1\ 3)(2\ 5\ 4)$

$\pi_1\pi_2\pi_3 = (1\ 2\ 5\ 4)(3)$. 
How to represent factorizations

For $k = 3$, $[n]$:  

$\pi_1 = (1)(2\ 3)(4)(5)$
$\pi_2 = (1\ 2)(3)(4)(5)$
$\pi_3 = (1\ 3)(2\ 5\ 4)$

$\pi_1 \pi_2 \pi_3 = (1\ 2\ 5\ 4)(3)$. 
How to represent factorizations

For $k = 3$, $[n]$:  

\[
\begin{array}{c}
\text{cycles } \pi_1: \quad \text{cycles } \pi_2: \quad \text{cycles } \pi_3:
\end{array}
\]

Examples

\[
\begin{align*}
\pi_1 &= (1)(2\ 3)(4)(5) \\
\pi_2 &= (1\ 2)(3)(4)(5) \\
\pi_3 &= (1\ 3)(2\ 5\ 4) \\
\pi_1\pi_2\pi_3 &= (1\ 2\ 5\ 4)(3).
\end{align*}
\]
How to represent factorizations

For \( k = 3 \), \([n]\): 

\[
\begin{array}{c}
\begin{array}{c}
1 \\
2 \\
\ldots \\
n
\end{array}
\end{array}
\]

cycles \( \pi_1 \): 

\[
\begin{array}{c}
\begin{array}{c}
\text{cycles} \quad \pi_2: \\
\text{cycles} \quad \pi_3:
\end{array}
\end{array}
\]

Examples

\[
\begin{align*}
\pi_1 &= (1)(2\,3)(4)(5) \\
\pi_2 &= (1\,2)(3)(4)(5) \\
\pi_3 &= (1\,3)(2\,5\,4) \\
\pi_1\pi_2\pi_3 &= (1\,2\,5\,4)(3).
\end{align*}
\]
How to represent factorizations

For $k = 3$, $[n]$:

cycles $\pi_1$:  

cycles $\pi_2$:  

cycles $\pi_3$:  

Examples

$\pi_1 = (1)(2\ 3)(4)(5)$  

$\pi_2 = (1\ 2)(3)(4)(5)$  

$\pi_3 = (1\ 3)(2\ 5\ 4)$  

$\pi_1 \pi_2 \pi_3 = (1\ 2\ 5\ 4)(3)$.  

$\pi_1 = (1\ 3\ 2\ 4)$  

$\pi_2 = (1\ 2)(3)(4)$  

$\pi_3 = (1\ 3)(2)(4)$  

$\pi_1 \pi_2 \pi_3 = (1\ 2\ 3\ 4)$.  

How to represent factorizations

For $k = 3$, $[n]$: 

$$\begin{align*}
\text{cycles } \pi_1: & \quad \text{cycles } \pi_2: & \quad \text{cycles } \pi_3: \\
\pi_1 &= (1)(2 \, 3)(4)(5) & \pi_2 &= (1 \, 2)(3)(4)(5) & \pi_3 &= (1 \, 3)(2 \, 5 \, 4) \\
\pi_1 \pi_2 \pi_3 &= (1 \, 2 \, 5 \, 4)(3) & \pi_1 \pi_2 \pi_3 &= (1 \, 3 \, 2 \, 4) & \pi_1 \pi_2 \pi_3 &= (1 \, 3 \, 2 \, 4) \\
\end{align*}$$
How to represent factorizations

For $k = 3$, $[n]$: 

cycles $\pi_1$: 

$\pi_1 = (1)(2\ 3)(4)(5)$

$\pi_2 = (1\ 2)(3)(4)(5)$

$\pi_3 = (1\ 3)(2\ 5\ 4)$

$\pi_1\pi_2\pi_3 = (1\ 2\ 5\ 4)(3)$. 

Examples 

$\pi_1 = (1\ 3\ 2\ 4)$

$\pi_2 = (1\ 2)(3)(4)$

$\pi_3 = (1\ 3)(2\ 4)$

$\pi_1\pi_2\pi_3 = (1\ 2\ 3\ 4)$. 

How to represent factorizations

For $k = 3$, $[n]$: \[1\quad 2\quad \ldots\quad n\]

Cycles $\pi_1$: \[\begin{array}{c}
\pi_1 = (1)(2\ 3)(4)(5) \\
\pi_2 = (1\ 2)(3)(4)(5) \\
\pi_3 = (1\ 3)(2\ 5\ 4)
\end{array}\]

Cycles $\pi_2$: \[\begin{array}{c}
\pi_1 = (1\ 2\ 5\ 4) \\
\pi_2 = (1\ 3\ 2\ 4) \\
\pi_3 = (1\ 3)(2\ 4)
\end{array}\]

Cycles $\pi_3$: \[\begin{array}{c}
\pi_1 = (1\ 2\ 3\ 4) \\
\pi_2 = (1\ 2)(3)(4) \\
\pi_3 = (1\ 3)(2)(4)
\end{array}\]

Examples:

\[
\begin{array}{c}
\pi_1 = (1\ 3\ 2\ 4) \\
\pi_2 = (1\ 2)(3)(4) \\
\pi_3 = (1\ 3)(2\ 4)
\end{array}
\]
How to represent factorizations

For $k = 3$, $[n]$: \[ \begin{array}{ccc} 1 & \ldots & n \end{array} \]

cycles $\pi_1$: \quad cycles $\pi_2$: \quad cycles $\pi_3$:

Examples

\[
\begin{align*}
\pi_1 &= (1)(2\,3)(4)(5) \\
\pi_2 &= (1\,2)(3)(4)(5) \\
\pi_3 &= (1\,3)(2\,5\,4) \\
\pi_1\pi_2\pi_3 &= (1\,2\,5\,4)(3).
\end{align*}
\]
How to represent factorizations

For $k = 3$, $[n]$:  

cycles $\pi_1$:  

cycles $\pi_2$:  

cycles $\pi_3$:  

Examples

$\pi_1 = (1)(2 \ 3)(4)(5)$  
$\pi_2 = (1 \ 2)(3)(4)(5)$  
$\pi_3 = (1 \ 3)(2 \ 5 \ 4)$

$\pi_1 \pi_2 \pi_3 = (1 \ 2 \ 5 \ 4)(3)$.  

$\pi_1 = (1 \ 3 \ 2 \ 4)$  
$\pi_2 = (1 \ 2)(3)(4)$  
$\pi_3 = (1 \ 3)(2)(4)$

$\pi_1 \pi_2 \pi_3 = (1 \ 2 \ 3 \ 4)$. 
How to represent factorizations

For $k = 3$, $[n]$: 

cycles $\pi_1$: 

$\pi_1 = (1)(23)(4)(5)$

cycles $\pi_2$: 

$\pi_2 = (12)(3)(4)(5)$

cycles $\pi_3$: 

$\pi_3 = (13)(254)$

Examples 

$\pi_1\pi_2\pi_3 = (1254)(3)$. 

$\pi_1 = (1324)$

$\pi_2 = (12)(3)(4)$

$\pi_3 = (13)(2)(4)$

$\pi_1\pi_2\pi_3 = (1234)$. 
Maps and constellations

A **map** is *cellular* embedding of connected graph on surface.

A **constellation** $C$ is map with gray and white faces:
- **gray faces** are $k$-gons (vertices 1, 2, $\ldots$, $k$), **edges** of type $i$: $(i, i+1)$.
  - **size** of constellation is $\#\{\text{gray faces}\}$,
  - **unicellular**: only one white face,
  - **rooted**: distinguished edge $(3, 1)$,
  - **genus**: $g$ given by equation:

$$v - e + f = 2 - 2g \rightarrow q_1 + \cdots + q_k = (k - 1)n + 1 - 2g.$$ 

**Examples**

- size 5, genus 0
- size 5, genus 0, unicellular
- size 4, genus 1, unicellular, rooted
Planar maps and constellations

planar case \((g = 0)\):

\[ q_1 + \cdots + q_k = (k - 1)n + 1 \]

\(k=2\)

plane bipartite trees:

\[ K_{q,n+1-q}^n = \frac{1}{n} \binom{n}{q} \binom{n}{q-1} \]

plane trees:

\[ Cat(n) = \frac{1}{n+1} \binom{2n}{n} \]

General \(k\)

plane cacti:

\[ K_{q_1,\ldots,q_k}^n = n^{k-1} \prod_{i=1}^{k} \frac{1}{q_i} \binom{n-1}{q_i} \]

[Goulden-Jackson 92]
Correspondence factorizations and constellations

\( \{(\pi_1, \ldots, \pi_k) \in K_{q_1, \ldots, q_k}^n \} \iff \{ \text{rooted unicellular } k\text{-constellations } C \} \),

- \( q_i \) cycles of \( \pi_i \) \( \iff \) \( q_i \) type \( i \) vertices of \( C \),
- cycle(s) of \( \pi_1 \circ \cdots \circ \pi_k \) \( \iff \) white face(s) of \( C \).

**Examples**

\[
\begin{align*}
\pi_1 &= (1 2)(3)(4)(5) \\
\pi_2 &= (1)(2 5)(3)(4) \\
\pi_3 &= (1)(2 3 4)(5) \\
\pi_1 \pi_2 \pi_3 &= (1 2 3 4 5).
\end{align*}
\]


\[
\left\{ \left( \pi_1, \ldots, \pi_k \right) \in K_{q_1, \ldots, q_k}^n \right\} \iff \left\{ \text{rooted unicellular } k\text{-constellations } C \right\},
\]
\[
q_i \text{ cycles of } \pi_i \iff q_i \text{ type } i \text{ vertices of } C,
\]
\[
\text{cycle(s) of } \pi_1 \circ \cdots \circ \pi_k \iff \text{white face(s) of } C.
\]

**Examples**

\[
\pi_1 = (1 2)(3)(4)(5)
\]
\[
\pi_2 = (1)(2 5)(3)(4)
\]
\[
\pi_3 = (1)(2 3 4)(5)
\]
\[
\pi_1 \pi_2 \pi_3 = (1 2 3 4 5).
\]

\[
\pi_1 = (1 3 2 4)
\]
\[
\pi_2 = (1 2)(3)(4)
\]
\[
\pi_3 = (1 3)(2)(4)
\]
\[
\pi_1 \pi_2 \pi_3 = (1 2 3 4).
\]
Correspondence factorizations and constellations

\{ (\pi_1, \ldots, \pi_k) \in \mathcal{K}_{q_1, \ldots, q_k}^n \} \Leftrightarrow \{ \text{rooted unicellular } k\text{-constellations } C \},

q_i \text{ cycles of } \pi_i

\text{cycle(s) of } \pi_1 \circ \cdots \circ \pi_k

\{ (\pi_1, \ldots, \pi_k) \in \mathcal{C}_{p_1, \ldots, p_k}^n \} \Leftrightarrow \{ \text{rooted unicellular } k\text{-constellations } C, \text{ size } n, \text{ type } i \text{ vertices colored using all colors } [p_i] \}

(\text{i.e. cycles } \pi_i \text{ colored using all colors } [p_i])

Examples

\pi_1 = (1 2)(3)(4)(5)
\pi_2 = (1)(2 5)(3)(4)
\pi_3 = (1)(2 3 4)(5)
\pi_1 \pi_2 \pi_3 = (1 2 3 4 5)
p_1 = p_2 = p_3 = 2

\pi_1 = (1 3 2 4)
\pi_2 = (1 2)(3)(4)
\pi_3 = (1 3)(2)(4)
\pi_1 \pi_2 \pi_3 = (1 2 3 4)
p_1 = 1, p_2 = p_3 = 2

This representation of factorizations as constellations is ideal to find bijections.
When coloring forces planarity

Since \# colors \leq \# vertices, if \( p_1 + \cdots + p_k = (k - 1)n + 1 \) then \( p_i = q_i \).

\( k = 2 \)
plane bipartite trees:

\[
K_{n, n+1-q}^n = \frac{1}{n} \binom{n}{q} \binom{n}{q-1}
\]

General \( k \)
plane cacti:

\[
K_{q_1, \ldots, q_k}^n = n^{k-1} \prod_{i=1}^{k} \frac{1}{q_i} \binom{n-1}{q_i}
\]

[Goulden-Jackson 92]
Outline of talk

I Background on Jackson’s formula ✓

II Symmetry of colored factorizations ✓

III Probabilistic puzzle equivalent to Jackson’s formula ✓

IV Bijections for symmetry and towards probabilistic puzzle

0. From colored factorizations to rooted colored unicellular ✓
   constellations,

1. From constellations to tree-rooted constellations,

(1)(2)(3) 0
(1)(2 3) ⇔
(1 3)(2) 1
Remark on rooted constellations

1. We can view the rooted unicellular constellation as a circuit on a polygon ($n \cdot k$-gon) with gluings of $n \cdot k$-gons.

Example
Remark on rooted constellations

1. We can view the rooted unicellular constellation as a circuit on a polygon (\(n \cdot k\)-gon) with gluings of \(n\ k\)-gons.

Example
1.1 Constellations to (induced graph + tour)

1. We can view the rooted unicellular constellation as a circuit on a polygon \((n \cdot k\)-gon\) with gluings of \(n k\)-gons.

Example

Summary: From \(C \in C_{p_1,\ldots,p_k}^n\) (rooted unicellular \(k\)-constellation, size \(n\), type \(i\) vertices colored using all colors in \([p_i]\)) obtain a rooted graph with \(p_i\) type \(i\) vertices + \(k\)-Eulerian tour \(\xi\).
1.2 (graph + Eulerian tour) to tree-rooted constellations

**BEST Theorem:** Let $\bar{G} = (V, A)$ connected digraph, each vertex has equal number of incoming and outgoing arcs, then

$$\{ \xi \mid \text{Eulerian tour starting/ending } \nu \} \iff \{ (T, O) \mid T \text{ spanning tree directed towards } \nu \text{ and } O \text{ ordering around each vertex of outgoing arcs not in } T \}$$

Back to example
1.2 (graph + Eulerian tour) to tree-rooted constellations

**BEST Theorem:** Let $\overline{G} = (V, A)$ connected digraph, each vertex has equal number of incoming and outgoing arcs, then

$$\left\{ \xi \mid \text{Eulerian tour starting/ending } \nu \right\} \iff \left\{ (T, O) \mid T \text{ spanning tree directed towards } \nu \right\}$$

**O** ordering around each vertex of outgoing arcs not in $T$

Back to example

- momentarily forget edge identifications,
1.2 (graph + Eulerian tour) to tree-rooted constellations

**BEST Theorem:** Let $\bar{G} = (V, A)$ connected digraph, each vertex has equal number of incoming and outgoing arcs, then

$$\{ \xi \mid \text{Eulerian tour starting/ending } v \} \Leftrightarrow \{ (T, O) \mid T \text{ spanning tree directed towards } v, O \text{ ordering around each vertex of outgoing arcs not in } T \}$$

**Back to example**

- momentarily forget edge identifications,
- apply BEST Theorem,
**BEST Theorem:** Let $\overline{G} = (V, A)$ connected digraph, each vertex has equal number of incoming and outgoing arcs, then

$$\{ \xi \mid \text{Eulerian tour starting/ending } v \} \iff \{ (T, O) \mid T \text{ spanning tree directed towards } v \}$$

**O** ordering around each vertex of outgoing arcs not in $T$.

**Back to example**

- momentarily forget edge identifications,
- apply BEST Theorem,
Secod remark on rooted constellations

2. Embedding Theorem: rooted constellations are in bijection with underlying graph $G + \text{rooted rotation system}$: i.e.
- total order of gray $k$-gons around a root vertex,
- cyclic order of gray $k$-gons around other vertices.

Example
**BEST Theorem:** Let $G = (V, A)$ be a connected directed graph such that each vertex has equal number of incoming and outgoing arcs, then
\[
\left\{ \xi \mid \text{Eulerian tour starting/ending } v \right\} \iff \left\{ (T, O) \mid T \text{ spanning tree directed towards } v, O \text{ ordering around each vertex of outgoing arcs not in } T \right\}
\]

Back to example

- momentarily forget edge identifications,
- apply BEST Theorem,
1.2 (graph + Eulerian tour) to tree-rooted constellations

**BEST Theorem:** Let $\overline{G} = (V, A)$ be a connected directed graph such that each vertex has equal number of incoming and outgoing arcs, then

$$\left\{ \xi \mid \text{Eulerian tour starting/ending } v \right\} \iff \left\{ (T, O) \mid T \text{ spanning tree directed towards } v, O \text{ ordering around each vertex of outgoing arcs not in } T \right\}$$

Back to example

- momentarily forget edge identifications,
- apply BEST Theorem,
- recall identifications and use **Embedding Theorem**.

rooted constellation
size 4
1 type 1 vertex,
2 type 2 vertices,
2 type 3 vertices.
1.2 (graph + Eulerian tour) to tree-rooted constellations

Another example

BEST Thm.
1.2 (graph + Eulerian tour) to tree-rooted constellations

Another example

Theorem [Bernardi-M 11]

\[
\begin{align*}
\{ & \text{rooted unicellular } k\text{-constellations, size } n, \text{ type } i \\
& \text{vertices colored using all colors in } [p_i]\} \\
\iff \\
\{ & \text{rooted } k\text{-constellations, size } n, \text{ with } p_i \text{ type } i \text{ vertices,} \\
& \text{with distinguished spanning tree oriented towards root vertex}\}
\end{align*}
\]

-Extends construction by [Bernardi 09], building up from [Lass 01].
1.2 (graph + Eulerian tour) to tree-rooted constellations

Another example

\[ C_{p_1, \ldots, p_k}^n \]

**Theorem [Bernardi-M 11]**

\[
\begin{align*}
&\text{rooted unicellular } k\text{-constellations, size } n, \text{ type } i \\
&\text{vertices colored using all colors in } [p_i]
\end{align*}
\]
\[\iff\]
\[
\begin{align*}
&\text{rooted } k\text{-constellations, size } n, \text{ with } p_i \text{ type } i \text{ vertices,} \\
&\text{with distinguished spanning tree oriented towards root vertex}
\end{align*}
\]

- Extends construction by [Bernardi 09], building up from [Lass 01].
1.2 (graph + Eulerian tour) to tree-rooted constellations

Another example

\[
\begin{align*}
\mathcal{C}^n_{p_1, \ldots, p_k} & \quad \iff \\
\text{tree-rooted constellations } (C^\bullet, T)
\end{align*}
\]

Theorem [Bernardi-M 11]

\[
\left\{ \begin{aligned}
\text{rooted unicellular } k\text{-constellations, size } n, \text{ type } i \\
\text{vertices colored using all colors in } [p_i]
\end{aligned} \right\} \iff \\
\left\{ \begin{aligned}
\text{rooted } k\text{-constellations, size } n, \text{ with } p_i \text{ type } i \text{ vertices,} \\
\text{with distinguished spanning tree oriented towards root vertex}
\end{aligned} \right\}
\]

- Extends construction by [Bernardi 09], building up from [Lass 01].
- Bijection preserves type/degree.
Outline of talk

I Background on Jackson’s formula ✓

II Symmetry of colored factorizations ✓

III Probabilistic puzzle equivalent to Jackson’s formula ✓

IV Bijectons for symmetry and towards probabilistic puzzle

0. From colored factorizations to rooted colored unicellular constellations ✓
   1. From constellations to tree-rooted constellations ✓
   2. Symmetry of tree-rooted constellations.
Symmetry of tree-rooted constellations

Recall, for \( \alpha(i) \) composition \( n \):
\[
\begin{align*}
\{ & \text{factorizations } \pi_1 \cdots \pi_k = \gamma, \\
& \text{cycles of } \pi_i \text{ are colored, } \pi_i \text{ has } \alpha_j(i) \text{ elements colored } j \}
\end{align*}
\]
\[\Leftrightarrow\]
\[
\begin{align*}
\{ & \text{rooted unicellular } k\text{-constellations } C, \text{ size } n, \text{ has } \alpha_j(i) \text{ type } i \text{ vertices colored } j \\
& \text{tree-rooted } k\text{-constellation } (C^*, T), \text{ size } n, \text{ type } i \text{ vertices have degrees } \alpha(i) \}
\end{align*}
\]

\( C_{\alpha(1), \ldots, \alpha(k)} \)

Theorem [Bernardi-M 11]
\( \alpha(1), \ldots, \alpha(k) \) and \( \beta(1), \ldots, \beta(k) \) compositions of \( n \) with \( \ell(\alpha(i)) = \ell(\beta(i)) \) then \( \#T_{\alpha(1), \ldots, \alpha(k)} = \#T_{\beta(1), \ldots, \beta(k)} \).

Corollary
\( \alpha(1), \ldots, \alpha(k) \) and \( \beta(1), \ldots, \beta(k) \) compositions of \( n \) with \( \ell(\alpha(i)) = \ell(\beta(i)) \) then \( C_{\alpha(1), \ldots, \alpha(k)} = C_{\beta(1), \ldots, \beta(k)} \).
Symmetry of tree-rooted constellations

**Theorem [Bernardi-M 11]**

\( \alpha^{(1)}, \ldots, \alpha^{(k)} \) and \( \beta^{(1)}, \ldots, \beta^{(k)} \) compositions of \( n \) with \( \ell(\alpha^{(i)}) = \ell(\beta^{(i)}) \) then \( \#T_{\alpha^{(1)}, \ldots, \alpha^{(k)}} = \#T_{\beta^{(1)}, \ldots, \beta^{(k)}} \).

**Proof uses an involution**
Symmetry on planar maps and constellations

planar case ($g = 0$):

$\ell(\alpha^{(1)}) + \cdots + \ell(\alpha^{(k)}) = (k - 1)n + 1$

Enough to compute the **hook case** $\alpha^{(i)} = (1^{p_i-1}, n + 1 - p_i)$.

$k=2$

plane bipartite trees:

$k=2$

plane cacti:

plane trees:

General $k$

plane cacti:

$K^n_{q,n+1-q} = \frac{1}{n} \binom{n}{q} \binom{n}{q-1}$

$K^n_{q_1,\ldots,q_k} = n^{k-1} \prod_{i=1}^{k} \frac{1}{q_i} \binom{n-1}{q_i}$

[Goulden-Jackson 92]
Outline of talk

I Background on Jackson’s formula ✓

II Symmetry of colored factorizations ✓

III Probabilistic puzzle equivalent to Jackson’s formula ✓

IV Bijections for symmetry and towards probabilistic puzzle

0. From colored factorizations to rooted colored unicellular ✓ constellations,
   1. From constellations to tree-rooted constellations, ✓
   2. Symmetry of tree-rooted constellations. ✓
Bibliography:

- (with O. Bernardi) **Bijections and symmetries for the factorizations of the long cycle**, arXiv:1112.4970

- (with O. Bernardi) **Trees from sets: a probabilistic arborescence with algebraic roots** (in preparation)
From tree-rooted constellations to tree-rooted mobiles

Example

Idea: looking at dual \{tree-rooted constellations\} ⇔ \{unicellular map with inbuds, outbuds forming parenthesis system\}.

A blossoming mobile \(B\) is a bipartite unicellular map with inbuds and outbuds (same number), \(B\) is balanced if buds form parenthesis system (difficult condition)
Mobiles that are not necessarily balanced

\[ \mathcal{B}_{p_1, \ldots, p_k}^n := \{ \text{blossoming mobiles } n \text{ black vertices, edges and buds} \]
\[ \text{colored with } [k], \text{ rooted corner } (k, 1), p_i \text{ color } i \text{ in-buds/outbuds, \ldots, not necessarily balanced.} \]

\[ B_{p_1, \ldots, p_k}^n := \# \mathcal{B}_{p_1, \ldots, p_k}^n \]

Example

\[
\begin{align*}
\mathcal{B}_{p_1, \ldots, p_k}^n &\iff \text{tree-rooted constellations } (C^\bullet, T) \text{ where root of } C^\bullet \text{ not necessarily equal to root of } T \text{ (tree still oriented from vertex type } i \text{ to vertex type } i - 1) \\
\end{align*}
\]
Mobiles that are not necessarily balanced

\[ B^n_{p_1, \ldots, p_k} := \left\{ \begin{array}{l}
\text{blossoming mobiles } n \text{ black vertices, edges and buds}
\end{array} \right. \]
\begin{array}{l}
\text{colored with } [k], \text{ rooted corner } (k, 1), \ p_i \text{ color } i \text{ in-buds/outbuds, \ldots, not necessarily balanced.}
\end{array} \]

\[ B^n_{p_1, \ldots, p_k} := \# B^n_{p_1, \ldots, p_k} \]

Example

Lemma:

\[ B^n_{p_1, \ldots, p_k} \iff \left\{ \begin{array}{l}
\text{tree-rooted constellations } (C^\bullet, T) \text{ where root of } C^\bullet \text{ not}
\end{array} \right. \]
\begin{array}{l}
\text{necessarily equal to root of } T \text{ (tree still oriented from}
\end{array} \]
\begin{array}{l}
\text{vertex type } i \text{ to vertex type } i - 1)
\end{array} \]
Mobiles that are not necessarily balanced

\[ B^n_{p_1, \ldots, p_k} := \begin{cases} \text{blossoming mobiles } n \text{ black vertices, edges and buds} \\ \text{colored with } [k], \text{ rooted corner } (k, 1), \text{ } p_i \text{ color } i \text{ in-} \\ \text{buds/outbuds, } \ldots, \text{not necessarily balanced.} \end{cases} \]

\[ B^n_{p_1, \ldots, p_k} := \# B^n_{p_1, \ldots, p_k} \]

Example

Lemma:

\[ B^n_{p_1, \ldots, p_k} \Leftrightarrow \begin{cases} \text{tree-rooted constellations } (C^\bullet, T) \text{ where root of } C^\bullet \text{ not} \\ \text{necessarily equal to root of } T \text{ (tree still oriented from} \\ \text{vertex type } i \text{ to vertex type } i - 1) \end{cases} \]

Theorem [Bernardi-M]:

\[ B^n_{p_1, \ldots, p_k} \Leftrightarrow (\prod_{i=1}^k p_i! : 1) \]

\[ C^n_{p_1+1, p_2, \ldots, p_k} \cup C^n_{p_1, p_2+1, \ldots, p_k} \cup \cdots \cup C^n_{p_1, p_2, \ldots, p_k+1}. \]
From blossoming mobiles to Eulerian structures

Example $k = 3, n = 4$

1. Start with blossoming mobile $B \in B_{p_1, \ldots, p_k}^n$.
2. Choose arbitrary pairing of inbuds and outbuds of the same type ($\prod p_i!$).
3. Number black vertices from appearance of ingoing 1 edges around face.
1. Start with blossoming mobile $B \in B_{p_1,\ldots,p_k}^n$.
2. Choose arbitrary pairing of inbuds and outbuds of the same type $(\prod p_i !)$.
3. Number black vertices from appearance of ingoing 1 edges around face.
4. For $i \in [n], j \in [k], i \in S_j$ iff type $j$ edge around $i$th black vertex is a bud.
5. For $j \in [k] \sigma_j \in \mathfrak{S}_n$ is order of appearance of type $j$ edges around the face.

The tour of the face of $B$ induces an Eulerian tour on $K_{k,n}$.
From blossoming mobiles to Eulerian structures (cont.)

Example \( k = 3, n = 4 \)

\[ \begin{align*}
\sigma_1 &= 1234 \\
\sigma_2 &= 4321 \\
\sigma_3 &= 4132.
\end{align*} \]

A pair \( ((S_1, \ldots, S_k), (\sigma_1, \ldots, \sigma_k)) \), where \( (S_1, \ldots, S_k) \in \mathcal{M}_{p_1, \ldots, p_k}^n \) and \( \sigma_i \in \mathcal{S}_n \) defines a tour structure on \( K_{k,n} \).

The tour structure is **Eulerian** if the tour is Eulerian.

**Theorem [Bernardi-M]:**

\[ \mathcal{B}_{p_1, \ldots, p_k}^n \stackrel{(1 : \prod p_i!)}{\leftrightarrow} \{ \text{Eulerian tour structures } ((S_1, \ldots, S_k), (\sigma_1, \ldots, \sigma_k)) \} \]