3.3.9. (a) The equipotentials \( u(x, y) = y \) are horizontal lines, hence the streamlines should be vertical lines: \( s(x, y) = x \). We have \( \nabla u \cdot \nabla s = (0, 1) \cdot (1, 0) = 0 \).

(b) \( u(x, y) = x - y = c \) are 45 degree lines, hence the streamlines should be 135 degree lines: \( s(x, y) = x + y \). We have \( \nabla u \cdot \nabla s = (1, -1) \cdot (1, 1) = 0 \).

(c) \( u(x, y) = \log((x^2 + y^2)^{1/2}) = c \) are concentric circles centered at the origin, hence the streamlines should be lines passing through the origin: \( s(x, y) = y/x \) (or any other function of \( y/x \)). We have

\[
\nabla u \cdot \nabla s = \left( \frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right) \cdot \left( \frac{-y}{x^2}, \frac{1}{x} \right) = 0.
\]

3.3.10. We have: \( (u_x, u_y) = (2xy, x^2 - y^2) \). Comparing coordinates, we obtain \( u = x^2 y + F(y) \) and \( u = x^2 y - \frac{1}{3} y^3 + G(x) \), therefore \( u(x, y) = x^2 y - \frac{1}{3} y^3 \). For \( s \), we have \( (s_y, -s_x) = (2xy, x^2 - y^2) \) which are the same equations as for \( u \) before, except with \( x, y \) switched. Thus \( s(x, y) = xy^2 - \frac{1}{3} x^3 \). The upper boundary \( y = x/\sqrt{3} \) (which is also \( s(x, y) = 0 \)) has \( n = (\cos \frac{2\pi}{3}, \sin \frac{2\pi}{3}) = \left( -\frac{1}{2}, \frac{\sqrt{3}}{2} \right) \). Therefore,

\[
v \cdot n = \frac{1}{2} (2xy, x^2 - y^2) \cdot (-1, \sqrt{3}) = \frac{1}{2} \left( 2x^2/\sqrt{3}, x^2 - x^2/3 \right) \cdot (-1, \sqrt{3}) = 0.
\]

3.4.2. Since the Laplacian of \( x^2 + y^2 \) is 4, we just need to adjust the function so that it vanishes on the circle \( x^2 + y^2 = 1 \). Thus, \( u(x, y) = x^2 + y^2 - 1 \) works.

3.4.4. We compute \( u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta \theta} = \frac{1}{r^2} \cos \theta + \frac{\cos \theta}{r} - \frac{\cos \theta}{r^2} - \frac{\cos \theta}{r^2} - \frac{\cos \theta}{r^2} = 0 \), thus \( u \) satisfies Laplace’s equation (13). To express \( u \) in terms of \( x, y \), recall that \( x = r \cos \theta, y = r \sin \theta \). Hence

\[
u = x + \frac{x}{x^2 + y^2}, (u_x, u_y) = \left( 1 + \frac{y^2 - x^2}{(x^2 + y^2)^2}, \frac{-2xy}{(x^2 + y^2)^2} \right)
\]

At a point \( (x, y) \) on the unit circle \( x^2 + y^2 = 1 \), the unit normal vector is \( n = (x, y) \), whereas \( v = (u_x, u_y) = (1 + y^2 - x^2, -2xy) \), hence \( v \cdot n = x + x(y^2 - x^2 - 2y^2) = x(1 - x^2 - y^2) = 0 \).

3.4.5. For \( u = \log r, u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta \theta} = -\frac{1}{r^2} - \frac{1}{r} + \frac{1}{r} + 0 = 0 \). From problem 3.3.9(c), we know that the streamlines are lines passing through the origin, which are parameterized by their angle \( \theta \): \( s = \theta = \tan^{-1} \frac{y}{x} \). Hence \( u + is = \log(r e^{i\theta}) = \log z \). Since \( U = \log r^2 = 2u \), it also satisfies the Laplace equation, and \( \nabla \cdot 2s = 2\theta \). We check the Cauchy-Riemann equations for \( u, s \):

\[
u = \frac{1}{2} \log(x^2 + y^2) \Rightarrow (u_x, u_y) = \left( \frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right)
\]

\[
s = \tan^{-1} \frac{y}{x} \Rightarrow (s_x, s_y) = \left( \frac{-y}{x^2}, \frac{1}{1 + (y/x)^2} \right)
\]

3.5.5. The molecule should have the form

<table>
<thead>
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<th>-1</th>
<th>-4</th>
<th>-1</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4</td>
<td>20</td>
<td>-4</td>
</tr>
<tr>
<td>-1</td>
<td>-4</td>
<td>-1</td>
</tr>
</tbody>
</table>

The bandwidth of \( K2D \) is \( N \).

3.5.8. The \( ij \) block of \( C = \text{kron}(A, B) \) is the \( n \times n \) matrix \( A_{ij} B \). Hence the \( ij \) block of \( C^T \) is \( A_{ij} B^T \), i.e. \( C^T = \text{kron}(A^T, B^T) \). We now check that \( \text{kron}(A, B)\text{kron}(A^{-1}, B^{-1}) = I \):

The \( ij \) block of the above product is

1
\[ A_{i1}B(A^{-1})_{1j}B^{-1} + \ldots + A_{in}B(A^{-1})_{nj}B^{-1} = (A_{i1}(A^{-1})_{1j} + \ldots + A_{in}(A^{-1})_{nj}) BB^{-1} = \delta_{ij}I_n \]

Thus the diagonal blocks are \( I_n \), and the off-diagonal blocks are 0, i.e. \( \text{kron}(A,B)\text{kron}(A^{-1},B^{-1}) = I_n^{n^2} \).

**Matlab Problem**

We have \( N = 7 \), and \( f \) is an all-ones vector of dimension \( N^2 = 49 \). We need to find \( u = (K2D)^{-1}f \), where \( K2D = \text{kron}(I_7,K_7) + \text{kron}(K_7,I_7) \). The Matlab code is:

```matlab
K = toeplitz([2 -1 zeros(1,5)]);
I = eye(7);
f = ones(49,1);
K2D = kron(I,K)+kron(K,I);
u = K2D \ f;
u(25);
```

The value at the center of the grid should be \( u(25) = 4.6581 \).