18.085 PROBLEM SET 7 SOLUTIONS

HAOFEI WEI

Problem (3.2.4).

Solution. For the differential equation $u''' = \delta(x)$, we know from (1) that the solution is of the form

$$u(x) = \begin{cases} A + Bx + Cx^2 + \left(D - \frac{1}{6}\right)x^3 & \text{for } x \leq 0 \\ A + Bx + Cx^2 + Dx^3 & \text{for } x \geq 0 \end{cases}$$

We must fit the conditions $u = u'' = 0$ at $x = 1$ and $x = -1$. Thus, the equations we must satisfy are

\[
\begin{align*}
    u(-1) &= A - B + C - D + \frac{1}{6} = 0 \\
    u(1) &= A + B + C + D = 0 \\
    u''(-1) &= 2C - 6D + 1 = 0 \\
    u''(1) &= 2C + 6D = 0
\end{align*}
\]

Using matrices, we can express these equations as

\[
\begin{bmatrix}
1 & -1 & 1 & -1 \\
1 & 1 & 1 & 1 \\
0 & 0 & 2 & -6 \\
0 & 0 & 2 & 6
\end{bmatrix}
\begin{bmatrix}
A \\
B \\
C \\
D
\end{bmatrix}
= \begin{bmatrix}
\frac{-1}{6} \\
0 \\
1 \\
0
\end{bmatrix}
\]

Solving this equation, we find that $[A \ B \ C \ D] = \frac{1}{12} \begin{bmatrix} 2 \\ 0 \\ -3 \\ 1 \end{bmatrix}$.

\[
\square
\]

Problem (3.2.6).

Solution. (1) For $u = 0$ at both ends, we see that only $\phi_0^d$ and $\phi_3^d$ do not satisfy this. For $u' = 0$ at both ends, only $\phi_0^d$ and $\phi_3^d$ do not satisfy this. Thus, the functions we must drop are $\phi_0^d$, $\phi_3^d$, $\phi_0^s$, and $\phi_3^s$.

(2) For $u = 0$ at both ends, only $\phi_0^d$ and $\phi_3^d$ do not satisfy the condition. The $u'' = 0$ condition is not essential, and so the only functions to be dropped are $\phi_0^d$ and $\phi_3^d$.

(3) For $u(0) = u'(0) = 0$, $\phi_0^d$ and $\phi_3^d$ do not satisfy the condition. $u''(1) = u'''(1) = 0$ are not essential boundary conditions, so the only functions to be dropped are $\phi_0^d$ and $\phi_0^s$.

\[
\square
\]
Problem (3.2.8).

Solution. From figure 3.7 in the textbook, we see that $\phi$'s centered around $i\theta h$ are given by
$$\phi_i^d = \left(\frac{|x-ih|}{h} - 1\right)^2 \left(\frac{|x-ih|}{h} + 1\right)\quad \text{and} \quad \phi_i^s = \left(\frac{|x-ih|}{h} - 1\right)^2 \left(\frac{x-ih}{h}\right).$$

For $i=0$ and $h = \frac{1}{3}$, $\left|\frac{x}{h}\right| = 3x$ since $x$ ranges from 0 to $\frac{1}{3}$. Thus, our second derivatives simplify to $(\phi_0^d)'' = 324x - 54$ and $(\phi_0^s)'' = 162x - 36$.

For $i=1$ and $h = \frac{1}{3}$, $\left|\frac{x}{h}\right| = -3(x - \frac{1}{3})$ since $x - \frac{1}{3}$ ranges from $-\frac{1}{3}$ to 0. Thus, our two second derivatives simplify to $(\phi_1^d)'' = 54 - 324x$ and $(\phi_1^s)'' = 162x - 18$.

□

Problem (3.2.9).

Solution. Because the integral is order-independent, we know that $K_e$ must be symmetric.

Defining $\phi_{12} = (\phi_0^d)''(\phi_0^s)'$, we can calculate $K_{1,2} = K_{2,1}$ to be:

$$\int_0^{\frac{1}{3}} (\phi_0^d(x))''(\phi_0^s(x))''\,dx = \frac{1}{18}\phi_{12}(0) + \frac{4}{18}\phi_{12}(\frac{1}{6}) + \frac{1}{18}\phi_{12}(\frac{1}{3}) = \frac{1}{18}1944 + \frac{4}{8}0 + \frac{1}{18}972 = 162$$

Repeating this process for all other terms, we find that

$$K_e = \begin{bmatrix} 324 & 162 & -324 & -162 \\ 162 & 108 & -162 & +54 \\ -324 & -162 & 324 & -162 \\ -162 & +54 & -162 & 108 \end{bmatrix}.$$

□

Problem (3.2.10).

Solution. The $\phi$'s, when restricted to the given intervals, are just shifted in $x$ from each other by multiples of $\frac{1}{3}$, and so their integrals should be the same. Thus, $K_e$ should look the same for all three intervals. The combined $K$ is just the sum of these three $K_e$'s, with zeroes added in the proper places to indicate zero integrals, such as $\int_0^{\frac{1}{3}} \phi_2^d \phi_2^d$. Thus, our final matrix is

$$K = \begin{bmatrix} 324 & 162 & -324 & -162 & 0 & 0 & 0 & 0 \\ 162 & 108 & -162 & +54 & 0 & 0 & 0 & 0 \\ -324 & -162 & 648 & 0 & -324 & -162 & 0 & 0 \\ -162 & +54 & 0 & 216 & -162 & +54 & 0 & 0 \\ 0 & 0 & -324 & -162 & 648 & 0 & \textcolor{red}{-324} & \textcolor{red}{-162} \\ 0 & 0 & -162 & +54 & 0 & 216 & \textcolor{red}{-162} & \textcolor{red}{54} \\ 0 & 0 & 0 & 0 & -324 & -162 & 324 & 162 \\ 0 & 0 & 0 & 0 & -162 & +54 & 162 & 108 \end{bmatrix}.$$

□

Problem (3.2.11(a)).

Solution. To fit the boundary condition, we must remove $\phi_0^d, \phi_3^d, \phi_0^s$, and $\phi_3^s$ (see problem 3.2.6(a)). These functions correspond to rows and columns 1, 2, 7, and 8. Thus, we have
HAOFEI WEI

18.085 PROBLEM SET 7 SOLUTIONS

\[
K = \begin{bmatrix}
648 & 0 & -324 & -162 \\
0 & 216 & -162 & 54 \\
-324 & -162 & 648 & 0 \\
-162 & 54 & 0 & 216
\end{bmatrix}
\]

Using MatLab, we find that \(|K| \neq 0\), and so \(K\) is invertible.

\[\square\]

**Problem (3.3.1).**

**Solution.** \(v = (1, 0) = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y})\). Since \(\frac{\partial u}{\partial x} = 1\), we can integrate to find \(u(x, y) = x + f(y)\), where \(f(y)\) is a function of \(y\) only. Differentiating by \(y\), we can find \(\frac{\partial f(y)}{\partial y} = 0\), so \(f(y) = C\). Thus, we can find by inspection the potential \(u(x, y) = x + C\), and our equipotentials are along the lines \(x = \text{const.}\)

\(w = (1, 0) = (\frac{\partial s}{\partial y}, -\frac{\partial s}{\partial x})\). Through a similar process as above, we see that the streaming function is \(s(x, y) = y + D\), and our stream lines are \(y = \text{const.}\).

For \(w = (0, x) = (\frac{\partial s}{\partial y}, -\frac{\partial s}{\partial x})\). We can integrate the first term again to find \(s(x, y) = f(x)\).

\(-\frac{\partial s}{\partial x} = -\frac{\partial f(x)}{\partial x} = x\). Integrating, we find that \(f(x) = -\frac{x^2}{2} + C\) and \(s(x, y) = -\frac{x^2}{2} + C\), so our streamlines are where \(x^2\) is constant.

\[\square\]

**Problem (3.3.5).**

**Proof.** For \(v_1 = -y, v_2 = 0\), Stoke's law states \(\int_C -ydx = \int \int_R \frac{\partial v_1}{\partial y} dxdy = \int \int_R dxdy\), which is just the area of the region \(R\). Thus, the area of \(R\) is equal to \(-\int_C ydx\), as needed.

Given \(x = a \cos(t), y = b \sin(t)\), and \(0 \leq t \leq 2\pi\), we have

\[
\text{Area}(R) = -\int_C ydx
= -\int_0^{2\pi} b \sin(t) \frac{d(a \cos(t))}{dt} dt
= -\int_0^{2\pi} b \sin(t) \cdot (-a \sin(t)) dt
= \int_0^{2\pi} ab \sin^2(t) dt
= \pi ab
\]

This agrees with the usual expression for the area of an ellipse.

\[\square\]

**Problem (3.3.7).**

**Solution.** If \(w = (\frac{\partial s}{\partial y}, -\frac{\partial s}{\partial x})\), then \(\nabla \cdot w = \frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} = 0\). For \(w = (x^2, y^2)\), \(\nabla \cdot w = 2x + 2y \neq 0\) in general. Thus, \(w \neq (\frac{\partial s}{\partial y}, -\frac{\partial s}{\partial x})\) for any \(s(x, y)\).

For \(w = (y^2, x^2)\), \(\nabla \cdot w = 0 + 0 = 0\), so \(w = (\frac{\partial s}{\partial y}, -\frac{\partial s}{\partial x})\) for some \(s(x, y)\). To find \(s\), we first note that \(\frac{\partial s}{\partial y} = y^2\). Thus, \(s(x, y) = \frac{x^3}{3} + f(x)\). From this, we see that \(-\frac{\partial s}{\partial x} = -f'(x) = x^2\), so \(f(x) = -\frac{x^3}{3} + C\). Thus, our stream function must be \(s(x, y) = \frac{1}{3}(y^3 - x^3) + C\).
Problem (MatLab Question). Enter the matrix in question 11(a) of section 3.2. Find the right-hand-side that corresponds to a point load at $x = \frac{1}{3}$. Then solve the linear system with MatLab's backslash. What is the (numerical value of the) displacement at the endpoint $x = 1$?

Solution. The matrix from 3.3.11(a) is $K = \begin{bmatrix} 648 & 0 & -324 & -162 \\ 0 & 216 & -162 & 54 \\ -324 & -162 & 648 & 0 \\ -162 & 54 & 0 & 216 \end{bmatrix}$. In the equation $KU = F$, we have the definition $F_i = \int_0^1 f(x)V_i(x)dx$. Since $f(x) = \delta(x - \frac{1}{3})$ for a point load at $x = \frac{1}{3}$, and $V_i(x) = \phi_i(x)$, we get $F_i = \phi_i(\frac{1}{3})$. Thus, $F = \begin{bmatrix} \phi_1^d(1/3) \\ \phi_1^s(1/3) \\ \phi_2^d(1/3) \\ \phi_2^s(1/3) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Solving the system with "\" in MatLab gives us $U = \begin{bmatrix} 0.0028 \\ 0.0007 \\ 0.0016 \\ 0.0099 \end{bmatrix}$, and so our solution is $u(x) = 0.0028\phi_1^d(x) + 0.0007\phi_1^s(x) + 0.0025\phi_2^d(x) + 0.0034\phi_2^s(x)$. At $x = 1$, all $\phi$'s equal 0, so we have $u(1) = 0$, as expected since one of the boundary conditions for our built-in beam is $u = 0$ at $x = 0$ and $x = 1$.

At $x=1/3$, we would then have $u = 0.0028$, since the only nonzero function at that point is $\phi_1^d$. 

□