Problem (2.2.5a).

Solution. Substituting the given values of $u'_1$, $u'_2$, and $u'_3$ into $\frac{d}{dt} ||u(t)||^2$, we get

$$\frac{d}{dt} ||u(t)||^2 = 2 (u_1 u'_1 + u_2 u'_2 + u_3 u'_3)$$

$$= 2 (u_1 (cu_2 - bu_3) + u_2 (au_3 - cu_1) + u_3 (bu_1 - au_2))$$

$$= 2 (cu_1 u_2 - cu_1 u_2 + bu_1 u_3 - bu_1 u_3 + au_2 u_3 - au_2 u_3)$$

$$= 0$$

Thus, $||u(t)||^2$ does not change with time, so $||u(t)||^2 = ||u(0)||^2$.

□

Problem (2.2.6).

Solution. The trapezoidal rule for $u' = Au$, given by (24), is

$$\left( I - \frac{\Delta t}{2} A \right) U_{n+1} = \left( I + \frac{\Delta t}{2} A \right) U_n$$

Multiplying both sides by $U_{n+1}^T + U_n^T$ (since they're vectors), we get

$$\left( U_{n+1}^T + U_n^T \right) \left( I - \frac{\Delta t}{2} A \right) U_{n+1} = \left( U_{n+1}^T + U_n^T \right) \left( I + \frac{\Delta t}{2} A \right) U_n$$

$$U_{n+1}^T U_{n+1} + U_n^T U_{n+1} - U_n^T U_{n+1} - U_n^T U_n = \frac{\Delta t}{2} (U_{n+1} + U_n)^T A (U_{n+1} + U_n)$$

$$||U_{n+1}||^2 - ||U_n||^2 = \frac{\Delta t}{2} (U_{n+1} + U_n)^T A (U_{n+1} + U_n)$$

The right hand side is a scalar, and thus is equal to its transpose. However, $A^T = -A$, so we get $(U_{n+1} + U_n) A^T (U_{n+1} + U_n) = -(U_{n+1} + U_n)^T A (U_{n+1} + U_n) = 0$. Thus, $||U_{n+1}||^2 = ||U_n||^2$, as needed.

□

Problem (2.2.8).

Solution. Using the Forwards Euler method, we get

$$U_{n+1} = U_n + h V_n$$

$$V_{n+1} = V_n - h U_n$$

Thus,
\[ U_{n+1}^2 + V_{n+1}^2 = (U_n + hV_n)^2 + (V_n - hU_n)^2 \]
\[ = U_n^2 + 2hV_n U_n + h^2 V_n^2 + V_n^2 - 2hV_n U_n + h^2 U_n^2 \]
\[ = (1 + h^2)(U_n^2 + V_n^2) \]

With the Backwards Euler method,
\[ U_{n+1} = U_n + hV_{n+1} \]
\[ V_{n+1} = V_n - hU_{n+1} \]

To avoid crossing \( n + 1 \) and \( n \) terms of \( U \) and \( V \), we can rewrite the backwards Euler method as
\[ U_n = U_{n+1} - hV_{n+1} \]
\[ V_n = V_{n+1} + hU_{n+1} \]

This gives us
\[ U_n^2 + V_n^2 = (U_{n+1} - hV_{n+1})^2 + (V_{n+1} + hU_{n+1})^2 \]
\[ = U_{n+1}^2 + h^2 V_{n+1}^2 + V_{n+1}^2 + h^2 U_{n+1}^2 + 2hV_{n+1} U_{n+1} - 2hV_{n+1} U_{n+1} \]
\[ = (1 + h^2)(U_{n+1}^2 + V_{n+1}^2) \]

Thus, \( U_{n+1}^2 + V_{n+1}^2 = \frac{1}{1 + h^2}(U_n^2 + V_n^2) \), as needed.

Next, we calculate \((1+h^2)^{\frac{2\pi}{h}}\) for \( h = \frac{2}{32} \). A simple calculation with some calculator-like device will yield the answer to be approximately 3.355.

Next, we must take the limit of the expression as \( h \) goes to zero.
\[ \lim_{h \to 0} (1 + h^2)^{\frac{2\pi}{h}} = \exp \left[ \lim_{h \to 0} \ln \left(1 + h^2\right)^{\frac{2\pi}{h}} \right] \]
\[ = \exp \left[ \lim_{h \to 0} \frac{2\pi}{h} \ln (1 + h^2) \right] \]

Using L’hospital’s rule, we can reduce this to
\[ \lim_{h \to 0} (1 + h^2)^{\frac{2\pi}{h}} = \exp \left[ \lim_{h \to 0} \frac{\frac{d}{dh} \ln (1 + h^2)}{\frac{d}{dh} h} \right] \]
\[ = \exp \left[ \lim_{h \to 0} \frac{2h}{1 + h^2} \right] \]
\[ = e^0 = 1 \]

\( \square \)

**Problem (2.3.7).**

**Solution.** The coefficients \( \dot{u} = (C, D) \) satisfies the equation
\[ A^T A \begin{bmatrix} C \\ D \end{bmatrix} = A^T b \]
where \( A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \) and \( b \) is as defined. Performing the multiplication gives us
\[
\begin{bmatrix} 4 & 6 \\ 6 & 14 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}
\]

To solve this equation, we can either invert the matrix on the left hand side and multiply on both sides, or simply use MatLab’s equation solving capabilities, specifically the “\( \backslash \)” command.

Doing either one of these things will give you the result of \((C, D) = (3, -1)\). \(\square\)

**Problem** (2.3.8).

**Solution.** To find the projection \( p \) of \( b \) onto \( y = 3 - x \), we simply multiply to get \( p = A \hat{u} = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix} \).

It is simple to check that the points \((0, 3), (1, 2), (2, 1), (3, 0)\) lie on the line \( y = 3 - x \).

Finally, we can verify through direct multiplication that \( A^T e = A^T (b - p) = A^T \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = 0 \).

**Problem** (2.3.10).

**Solution.** Since we only have one variable \( C \), we have \( A = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \). To find the solution, we must solve the system \( A^T A C = A^T \begin{bmatrix} 4 \\ 1 \\ 0 \\ 1 \end{bmatrix} \). The solution for this is easily found to be \( 4C = 6 \) or \( C = \frac{3}{2} \). Thus, \( y = \frac{3}{2} \) is the horizontal line which best fits the system. \(\square\)

**Problem** (2.3.12).

**Solution.** For a third order polynomial, we will have \( A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \).
Now, we must solve the equation $A^T A \begin{bmatrix} C^* \\ D \\ E \end{bmatrix} = A^T b$, where $b = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$. Using MatLab, we can solve $\begin{bmatrix} C^* \\ D \\ E \end{bmatrix} = A^T A \backslash A^T b = \begin{bmatrix} 4 \\ -4 \\ 1 \end{bmatrix}$.

Fitting instead to a cubic, we will be solving the system $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \\ F \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 1 \end{bmatrix}$. This system can be solved exactly, and gives $\begin{bmatrix} C^* \\ D \\ E \\ F \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{bmatrix} \begin{bmatrix} 4 \\ -4 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 1 \\ 0 \end{bmatrix}$.

The error in this result is $e = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{bmatrix} \begin{bmatrix} 4 \\ -4 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 4 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

This is expected, as the solution we found is an exact solution of the system. □