4.1.1 (b) \( f(x) = |\sin x| \) is an even function, hence it has a cosine Fourier series. We have \( a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\sin x| \, dx = \frac{2}{\pi} \), and for \( k > 0 \),

\[
a_k = \frac{2}{\pi} \int_0^\pi \sin x \cos kx \, dx = \frac{1}{\pi} \int_0^\pi \sin (k + 1)x - \sin (k - 1)x \, dx = \begin{cases} 0 & \text{if } k \text{ is odd} \\ -\frac{4}{\pi(k^2 - 1)} & \text{otherwise} \end{cases}
\]

Thus

\[
|\sin x| = \frac{2}{\pi} - \frac{4}{\pi} \sum_{j=1}^{\infty} \frac{\cos 2jx}{(2j)^2 - 1}.
\]

(c) \( f(x) = x \) is odd, hence it has a sine Fourier series. We have

\[
b_k = \frac{2}{\pi} \int_0^\pi x \sin kx \, dx = \frac{2}{\pi k^2} \left( \sin kx - kx \cos kx \right) \bigg|_0^\pi = \frac{2(-1)^{k+1}}{k}
\]

hence

\[
x = \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k} \sin kx.
\]

4.1.3. \( f \) is even, hence \( b_k = 0 \). We compute \( a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx = \frac{1}{2} \), and for \( k > 0 \), \( a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos (kx) \, dx = \frac{2}{\pi} \int_0^{\pi/2} \cos kx \, dx = \frac{2}{\pi k} \sin \left( \frac{\pi k}{2} \right) \).

4.1.4. By the orthogonality of sines and cosines we have \( a_0 = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} f(x) \, dx \), and for \( k > 0 \), \( a_k = \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} f(x) \cos 2\pi kx \, dx, b_k = \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} f(x) \sin 2\pi kx \, dx, c_k = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} f(x) e^{-2\pi i kx/\pi} \, dx \).

4.1.10. The solution \( u(r, \theta) \) has the form

\[
u = a_0 + ra_1 \cos \theta + rb_1 \sin \theta + r^2 a_2 \cos 2\theta + r^2 b_2 \sin 2\theta + ...
\]

The coefficients \( a_i, b_j \) are precisely the Fourier series coefficients for the boundary function \( u(1, \theta) = \begin{cases} 1 & 0 < \theta < \pi \\ 0 & -\pi < \theta < 0 \end{cases} \). Observe that \( u(1, \theta) = \frac{1}{2} + \frac{1}{2} \text{sw}(\theta) \), where \( \text{sw}(x) \) is the standard square wave function. Hence, we know that \( a_0 = \frac{1}{2}, a_k = 0, b_k = \frac{2}{\pi k} (1 - \cos k\pi) \). We find \( u(0, 0) = a_0 = \frac{1}{2} \)

\[
u(r, \theta) = \frac{1}{2} + \frac{2}{\pi} \left( r \sin x + r^3 \sin \frac{3x}{3} + r^5 \sin \frac{5x}{5} + ... \right).
\]

4.1.13 (a) \( \int_{-\pi}^{\pi} |F(x)|^2 \, dx = \int_{-\pi/2}^{\pi/2} dx = \pi \).

(b) The complex Fourier coefficients \( c_k \) are easily found from the \( a_k \) 's in 4.1.3. (this is true since our function is even): \( c_0 = a_0 = \frac{1}{2}, c_k = \frac{1}{\pi k} \sin \left( \frac{\pi k}{2} \right) \).
(c) The energy identity for $F(x)$ becomes

$$
\int_{-\pi}^{\pi} |F(x)|^2 dx = 2\pi \sum_{k=-\infty}^{\infty} |c_k|^2
$$

$$
\pi = 2\pi \left( \frac{1}{2^2} + \frac{1}{\pi^2} + \frac{1}{3^2 \pi^2} + \frac{1}{5^2 \pi^2} + \frac{1}{7^2 \pi^2} + \ldots \right)
$$

$$
= \frac{\pi}{2} + 4 \pi \left( \frac{1}{3^2} + \frac{1}{5^2} + \ldots \right).
$$

When $h = 2\pi$, the square pulse $F(x)$ actually becomes the constant function $F(x) \equiv 1$, and therefore $c_0 = 1$ ($c_0$ is the constant function which best approximates $F$) and all other $c_k$’s are 0.

4.3.10. If $w$ is a 64th root of 1, then $w^2$ is a 32nd root of 1 and $\sqrt{w} = w^{1/2}$ is a 128th root of 1.

4.3.11. We claim that for any $n \geq 2$, the sum of the $n$th roots of unity is 0. Indeed, they are the roots of the polynomial $x^n - 1$, hence by Viète’s formula their sum equals the coefficient of $x$ in $x^n - 1$, which is 0.

4.3.12. By definition, we have

$$
c_{N-k} = \frac{1}{N} \sum_{j=0}^{N-1} f_j \overline{\omega}^{j(N-k)} = \frac{1}{N} \sum_{j=0}^{N-1} f_j \overline{\omega}^{jN-k} = \frac{1}{N} \sum_{j=0}^{N-1} f_j \overline{\omega}^{-jk} = \frac{1}{N} \sum_{j=0}^{N-1} f_j \overline{\omega}^{jk}
$$

($w$ is an $N$th root of 1, hence so is $\overline{w} = w^{-1}$, whence $\overline{w}^N = 1$). Taking conjugates of both sides and keeping in mind that $f_j = \overline{f}_j$, we arrive at the claimed identity:

$$
c_{N-k} = \frac{1}{N} \sum_{j=0}^{N-1} f_j \overline{\omega}^{jk} = c_k.
$$