TOWARDS THE CONSTRUCTION OF tmf

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I really appreciate the effort that everybody has put into this. I like the material. There is a lot of beautiful mathematics in it. But unfortunately, we never really managed to make it so that people could really learn it. I can clearly see that it was a lot of work...

There are a few things I was wanting to talk about. Watching the story of this obstruction theory come up over several lectures made me think that it would be worthwhile to give a short overview, and try to hit on what I think are key points and that didn’t quite come out in some of the talks.

What are we trying to do in constructing tmf? We have the following category

$$I := \text{Aff}/\mathcal{M}_{\text{ell}}$$

which consists of certain kinds of affine open subsets over the moduli stack of elliptic curves in the étale topology. We can think of this more colloquially as follows. An object of $I$ is a pair $(R, C)$, where $R$ is a ring, and $C$ is an elliptic curve over $R$. And a morphism $(R, C) \to (R', C')$ is a ring map $\phi : R \to R'$ together with an isomorphism $\phi^* C \cong C'$. But we don’t want to consider all such pairs, namely, the map $\text{Spec}(R) \to \mathcal{M}_{\text{ell}}$ induced by $C$ should be étale. One puts the étaleness condition because it implies that $R$ is Landweber exact (i.e., that the corresponding map $\text{Spec}(R) \to \mathcal{M}_{FG}$ is flat).

By applying the Landweber exact functor theorem, we get a functor from $I$ to the category of multiplicative homology theories. That functor sends $(R, C)$ to the Landweber exact theory associated to the formal completion of $C$. The lifting problem is expressed as

$$E_\infty\text{-ring spectra } \downarrow \quad \downarrow \quad \{\text{multiplicative homology theories}\}$$

(1)

$$I \quad \leftarrow \quad \mathcal{D}$$

We didn’t necessarily want to lift this to $E_\infty$-ring spectra. We would have been happy to make it to spectra. All we wanted was to rigidify this diagram so as to be able to take its inverse limit.

You can say this in the language of stacks but you can state our task in a more down-to-earth way and in fact we chose that language when I wrote about this in my ICM talk in Zurich because I thought that the language of stacks was too out there. It was even exotic in the algebraic geometry literature. At this level, stacks are just a convenient expository device.

Lifting to $E_\infty$-ring spectra might make the problem seem harder, but really it cuts the space of lifts down to a manageable size and actually makes the problem easier. Here, stacks first arose as a mainly expository device, but there is a real thing that made them essential.

We want to understand if there is a lift of the diagram (1), and whether any two of them are homotopy equivalent. In other words, the goal of our obstruction theory is to understand the realization space $r\mathcal{D}$ of this diagram. This is the nerve of the category of lifts, i.e., whose objects are lifts as above and whose morphisms are natural transformations which are homotopy equivalences on objects. We want to show that this space is nonempty and connected. It would be even better if that space turned out to be contractible but, unfortunately, that turns out to be not quite right. That’s where Jacob Lurie’s point of view improves things.
What you need is for any object $i \in I$ an $E_\infty$-ring spectrum $E$, and for any morphism $i \to j$ a map $E \to F$ of $E_\infty$-ring spectra. Thus, you are bound to encounter an invariant of the shape of the category $I$, as well as the homotopy groups of the mapping space $\text{Map}_{E_\infty}(E,F)$. Assuming for a moment that we already have an $E_\infty$-ring spectrum associated to each object $i$ of $I$, then we would get a diagram

$$I^{op} \times I \to \{\text{abelian groups}\}$$

$$(i,j) \mapsto \pi_* \text{Map}_{E_\infty}(E,F)$$

where $E$ and $F$ are the respective images of $i$ and $j$. At its crudest level, our obstruction theory will use the cohomology theory of these abelian group valued diagrams.

We’ve cheated a little bit by assuming that we already had lifts to $E_\infty$-ring spectra of the multiplicative homology theories associated to objects $i \in I$, but the obstruction theory for lifting multiplicative homology theories to $E_\infty$-ring spectra is similar to the obstruction theory we are dealing with.

There is a general pattern that any obstruction theory follows:

1. Make an algebraic approximation to your topological setup.
2. Build resolutions of whatever you are trying to study by pieces for which the algebraic approximation is exactly right.

Some key words that fall into category (1) are “derivations” and “$\psi$-$\theta$-algebras”. On the other hand, “resolution model categories” are the way of making sense of (2). They lead to obstruction groups involving derived functors of derivations, and Hochschild homology. The art of the game is to pose the right problem and find the right kind of algebraic approximation so that the obstruction groups can be handled (the best situation being of course when they all vanish).

In our case, we need an algebraic approximation to a map of $E_\infty$-ring spectra $E \xrightarrow{f} F$. Smashing both sides with $F$ gives $f' : F \wedge E \to F \wedge F \to F$, where the last map is the multiplication of $F$. Applying $\pi_*$ gives a map of $F_*$-algebras, $F_*E \to F_*$, which could be our algebraic approximation to an element of $f \in \pi_0 \text{Map}_{E_\infty}(E,F)$. Here is a clever trick that I learned from Bill Dwyer: to extend this and get approximations for elements of $\pi_* \text{Map}_{E_\infty}(E,F)$ based at $f$, note that a pointed map $S^t \to \text{Map}_{E_\infty}(E,F)$ is equivalent to a $E_\infty$-ring map $E \to FS^t$ over $F$, i.e., a diagram

$$E \xrightarrow{f^*} FS^t \xrightarrow{1} F$$

Here, $F^{S^t}$ is the cotensor of $F$ with the sphere, using the topological structure on the category of $E_\infty$-ring spectra. The underlying spectrum of $F^{S^t}$ is just the mapping spectrum from $S^t$ into $F$, and it is homotopy equivalent to $F \vee \Sigma^{-t} F$. As an $F_*$-algebra, we have $\pi_* F^{S^t} \cong F_*[\epsilon]/\epsilon^2$, where $\epsilon$ is in degree $-t$. Our algebraic approximation to an element in $\pi_* \text{Map}_{E_\infty}(E,F)$ thus becomes a commuting diagram of $F_*$-algebras

$$F_*E \xrightarrow{f_*[\epsilon]/\epsilon^2} F_*$$

The ring homomorphism in the top line is something of the form $f' + D(-)\epsilon$, where $D$ is an $F_*$-linear derivation $D : F_*E \to F_*$. The group of these derivations is $\text{Der}_{F_*}(F_*E,F_*)$, which is equivalent to $\text{Der}_{F_0}(F_0E,F_0)$. In general, if $f' : B \to A$ is an augmented $A$-algebra, there is an equivalence $\text{Der}_A(B,A) \simeq \text{Hom}_A(\Omega^1_{B/A},A) \simeq \text{Hom}_A(f^*\Omega^1_{A/B/A},A)$, where $\Omega^1_{B/A}$ is the $B$-module of relative Kähler differentials (and the last equivalence is just by adjunction). In this instance,
we may set $B = F_0 E$ and $A = F_0$, to obtain that this module of derivations is equivalent to $\text{Hom}_{F_0}(f^*\Omega^1_{E/F_0}, F_0)$.

This is the essential idea. If we were just working with $A_\infty$-ring spectra, this would be a sufficiently good algebraic approximation. The commutative case is a little more involved than the associative one, so a bit more structure needs to be added, such as Dyer-Lashof operations, but this extra stuff winds up “coming along for free”. The real point is to come to grip with these derivations.

When I was first thinking about these groups, it was one of these experiences where every day, I had to reinvent the wheel. I couldn’t hold the obstruction groups in my head in an organized manner. And this is where the language of stacks suddenly came in and clarified everything.

In our case, $E$ and $F$ are Landweber exact, which means that the associated maps from $U := \text{Spec} E_0$ and $V := \text{Spec} F_0$ to $\mathcal{M}_{FG}$ are flat. Then we can form the pullback $V \times_{\mathcal{M}_{FG}} U$ which encodes the smash product of $E$ and $F$:

$$V \times_{\mathcal{M}_{FG}} U \cong \text{Spec } F_0 E = \text{Spec } \pi_0(F \wedge E).$$

The language of stacks allows us to reorganize this picture so that $U$ and $V$ eventually just drop out. The map $E \rightarrow F$ induces a map $V \rightarrow U$ again denoted $f$, and so we get a diagram

$$\begin{array}{ccc}
\text{Spec}(F_0 E) & \cong & V \times_{\mathcal{M}_{FG}} U \\
\downarrow & & \downarrow f \\
U & \rightarrow & \mathcal{M}_{FG}
\end{array}$$

where $i$ is the map induced by $f^* : \pi_0(F \wedge E) \rightarrow \pi_0(F)$. Our module of derivations can be written more geometrically as $\text{Hom}(i^*\Omega^1_{V \times_{\mathcal{M}_{FG}} U/V}, \mathcal{O}_V)$. Since this is a pullback square, these relative differentials are just pulled back from the differentials on $U$ relative $\mathcal{M}_{FG}$. In other words, our group of derivations is equivalent to $\text{Hom}(f^*\Omega^1_{U/\mathcal{M}_{FG}}, \mathcal{O}_V)$.

These maps to the moduli stack of formal groups factor through the moduli stack of elliptic curves, and we can write this picture as

$$\begin{array}{ccc}
V \times_{\mathcal{M}_{FG}} U & \rightarrow & U \\
\downarrow f & & \downarrow c_E \\
V & \rightarrow & \mathcal{M}_{\text{ell}}
\end{array}$$

Since $C_E$ is an étale elliptic curve, the map $C_E$ is étale. Consequently, we have an equivalence $\Omega^1_U/\mathcal{M}_{FG} \cong C_E^*\Omega^1_{\mathcal{M}_{\text{ell}}/\mathcal{M}_{FG}}$ (since the relative differentials of an étale map are trivial). The composite $C_E \circ f$ is equal to $C_F$, the map classifying the elliptic curve associated to our elliptic cohomology theory $F$. Thus, the sequence of pullbacks $f^*C_E^*\Omega^1_{U/\mathcal{M}_{\text{ell}}}$ is equivalent to $C_F^*\Omega^1_{U/\mathcal{M}_{\text{ell}}}$. This presents our group of derivations as $\text{Hom}(\Omega^1_{\mathcal{M}_{\text{ell}}/\mathcal{M}_{FG}}, \mathcal{O}_{\mathcal{M}_{\text{ell}}})$ restricted to the subset $C_F : V \rightarrow \mathcal{M}_{\text{ell}}$, which is great because $U$ and $V$ have now really fallen out of the picture.

Thus, in the business end of the obstruction theory, the $I^{op} \times I$ diagram cohomology is just

$$H^*(\mathcal{M}_{\text{ell}}, (\Omega^1_{\mathcal{M}_{\text{ell}}/\mathcal{M}_{FG}})^{\vee} \otimes \omega^t),$$

where I’ve put back a power of the canonical sheaf $\omega$ because the derivation we started with actually had degree $-t$. Note: In the obstruction theory of Dwyer, Kan, Smith, this is called a centric diagram, which is why the “twisted-arrow” or Hochschild-Mitchell cohomology reduces to this.
This whole story we talked about is similar to something that you encounter when studying basic abstract differential geometry. Namely, for $M$ a smooth manifold and $U$ an open subset, then you can define vector fields on $U$ as derivations from functions on $U$ to functions on $U$:

$$\text{Vect}(U) = \text{Der}(\mathcal{O}_U, \mathcal{O}_U).$$

If now $V \subset U$ is an open subset of $U$, it is intuitively obvious that you can restrict a vector field. But how do you restrict a derivation? From the point of view of derivations, it is not obvious that one can restrict them:

$$\mathcal{O}_U \xrightarrow{D} \mathcal{O}_U$$

They seem to rather be bivariant. The way to solve this problem is to note that $\text{Der}(\mathcal{O}_U, \mathcal{O}_U) \cong \text{Hom}_{\mathcal{O}_U}(\Omega^1_U, \mathcal{O}_U)$ and that for an étale map $i : V \to U$, we have an equivalence $\Omega^1_V \cong i^*\Omega^1_U$. This allows us to define restriction as the composite

$$\text{Hom}_{\mathcal{O}_U}(\Omega^1_U, \mathcal{O}_U) \to \text{Hom}_V(i^*\Omega^1_U, i^*\mathcal{O}_U) \cong \text{Hom}_V(\Omega^1_V, \mathcal{O}_V).$$

Summarizing, if you define vector fields to be derivations, the thing that makes them into a sheaf is a very special property: it is the isomorphism $\Omega^1_V \cong i^*\Omega^1_U$, which comes from the fact that the map is étale. The fact that our obstruction groups could be rephrased as [some variant of] (2) is a consequence of that very same property.

Now let us think about how to analyze the obstruction groups (2), which can also be rewritten as

$$\text{RHom}(\Omega^1_{\mathcal{M}_{\text{ell}}/\mathcal{M}_{\text{FG}}}, \omega^t).$$

Reducing mod $p$, we can express the relative cotangent complex $\Omega^1_{\mathcal{M}_{\text{ell}}/\mathcal{M}_{\text{FG}}} \otimes \mathbb{F}_p$ as the pushforward $i_*\Omega^1_{\mathcal{M}_{\text{ell}}}$ of the cotangent space of the ordinary locus, where $i$ denotes the inclusion $\mathcal{M}_{\text{ord}} \hookrightarrow \mathcal{M}_{\text{ell}}$.

Since locally, $\mathcal{M}_{\text{ord}}$ is obtained from $\mathcal{M}_{\text{ell}}$ by inverting one element, the above group is similar to $\text{Ext}_{\mathbb{Z}[x]}(\mathbb{Z}[x^\pm 1], \mathbb{Z}[x])$, which has a somewhat big and ugly $\text{Ext}^1$. So it is a little bit cumbersome to deal with (3). In order to handle the structure of $\Omega^1_{\mathcal{M}_{\text{ell}}/\mathcal{M}_{\text{FG}}}$, it is cleaner to break the problem into two calculations, one over $\mathcal{M}_{\text{ord}}$, and another one over $\mathcal{M}_{\text{ell}}$.

Recall that our problem is to find a lift in the following diagram:

Here, we can break up the realization problems into pieces by functorially putting every spectrum we encounter into a Hasse pullback square. Geometrically, this corresponds to using the stratification of the moduli stack $\mathcal{M}_{\text{ell}}$ in terms of ordinary and supersingular loci:

$$\mathcal{O}^{\text{top}} \xrightarrow{L_K(2)\mathcal{O}^{\text{top}}} L_K(2)\mathcal{O}^{\text{top}}$$

Note that this is still a diagram of sheaves on the whole stack $\mathcal{M}_{\text{ell}}$. We can fracture the problem of understanding the space of lifts into understanding the space of three compatible things:

a. the moduli space of possible $L_K(2)\mathcal{O}^{\text{top}}$,

b. the moduli space of possible $L_K(1)\mathcal{O}^{\text{top}}$, and
c. the moduli space of maps $L_{K(1)} O^\text{top} \to L_{K(1)} L_{K(2)} O^\text{top}$. The moduli space of possible $O^\text{top}$ is then a pullback of the first two moduli spaces over the third moduli space. We will use different obstruction theories to analyze each one of the above problems. For $a$, we use that the appropriate Der groups are zero, which gives us that for every supersingular elliptic curve we can realize in a functorial way its universal deformation, along with the action of the automorphism group of the curve. This, along with the fact that the Hasse invariant has distinct zeros allows us to construct a sheaf of $E_\infty$-spectra over $\overline{M}_\text{ell}$, supported on $M^\text{ordss}_\text{ell}$. For $b$ and $c$, we use the $K(1)$-local obstruction theory. The obstruction groups become $H^*(M^\text{ordss}_\text{ell}, \text{Der}_{θ,∞})$ and these are zero too, at least at odd primes. At the prime 2, the ordinary locus does have some cohomology, but we can change our algebraic approximation by using $KO$ instead of $K$, rewrite the algebra differently and, that way, you also don’t encounter any obstruction groups. Note that we are not changing the moduli stack: all the sheaves are on $\overline{M}_\text{ell}$. It turns out however, that the obstruction groups can be calculated on the stacks $M^\text{ord}_\text{ell}$ and $M^\text{ss}_\text{ell}$.

Let us go back to the construction of the string orientation. Recall that we have the following diagram

$$
\begin{array}{ccc}
\Sigma^{-1}\text{bstring} & \to & gl_1(S)
\\
\downarrow
\\
C & \to & \text{bstring},
\end{array}
$$

and that the string orientation is equivalent to having a map $C \to gl_1(tmf)$. Here, $gl_1(S)$ is the infinite delooping of $GL_1$ of the sphere spectrum, and $C$ is the mapping cone of the map from $\Sigma^{-1}\text{bstring}$. The set of such maps, if non-empty, is a torsor over $[\text{bstring}, gl_1(tmf)]$.

Because of the particular homotopy type of the spectra $C$ and $gl_1(tmf)$, the dotted map in the above diagram is completely determined by what it does on rational homotopy groups, which allows us to relate a map like this to the classical theory of characteristic series. This is quite remarkable, because such a map actually corresponds to an $E_\infty$ map $M\text{String} \to tmf$, wheras, to specify the characteristic series, you just need to write down a sequence of numbers. Roughly speaking, both $C$ and $gl_1(tmf)$ “look like K-theory” (or can be localized to look like K-theory), and maps from the $K$-theory spectrum to itself are determined by their effect on homotopy groups, so you can identify the $[\text{bstring}, gl_1(tmf)]$-torsor of above maps by looking at what this does on rational homotopy group.

For simplicity, we treat the complex version of the above diagram. Let us also put an arbitrary $E_\infty$-spectrum $R$ in the place of $tmf$:

$$
\begin{array}{ccc}
\Sigma^{-1}\text{bu} & \to & gl_1(S)
\\
\downarrow
\\
C & \to & \text{bu},
\end{array}
$$

(4)

Note that, since $π_0(gl_1(S)) = \mathbb{Z}$ and $π_{≥1}(gl_1(S)) = π_{≥1}(S)$, all the homotopy groups of $gl_1(S)$ are finite, and it is thus rationally trivial. It follows that $C$ is rationally homotopy equivalent to $bu$, which is why one gets a map $bu \to gl_1(R)\mathbb{Q}$ from the “Thom class” $\mathfrak{u}$. For $n ≥ 1$, let $b_n ∈ π_{2n}(gl_1(R))\mathbb{Q} = π_{2n} R \otimes \mathbb{Q}$ be the images of the Bott generators in $π_{2n}(bu)$. In all the cases of interest (i.e., $R = KO$ or $tmf$ [in which case, one would have to replace $bu$ by $bspin$ or $bstring$, respectively] or, more generally, any spectrum that represents a cohomology theory whose job in life is to really be a cohomolgy theory, i.e., anything that is build in a simple way out of Lanweber exact theories; not something like the image of $J$-spectrum), the map $\mathfrak{u}$ is detected rationally and thus completely determined by the sequence of $b_n$’s.

We now relate this to the classical theory of characteristic series. As explained earlier, the map $\mathfrak{u}$ corresponds to an $E_∞$-map $U : MU \to R$. After rationalizing $R$, we get two $E_∞$-maps from $MU$:...
the first one is the rationalization of $U$, and the second is the map that factors through rational homology. These two maps are typically not equal to each other, and the following diagram

\[ \begin{array}{ccc}
MU & \xrightarrow{U} & R \\
& \downarrow & \downarrow \\
HQ & \xrightarrow{} & R_Q
\end{array} \]

is therefore not commutative. The ratio of these two Thom classes, i.e., the failure of the above diagram to commute is a unit in the rational $R$-cohomology of $(\mathbb{Z} \times BU)^\times$. In other words, it is a stable exponential characteristic class.

By the splitting principle, the above class is determined by its image $K_\Phi(z) \in R_Q^*[[z]]$ under the map

\[ R_Q^0((\mathbb{Z} \times BU)^\times) \to R_Q^0(\mathbb{CP}^\infty) \subset (R_\ast \otimes \mathbb{Q}[[z]])^\times \]

coming from the universal line bundle $\mathbb{CP}^\infty \to (\mathbb{Z} \times BU)^\times$. Thinking of $U$ as a genus, the Hirzebruch characteristic series $K_\Phi(z)$ is the Chern character of that class, i.e., its value on the universal line bundle over $\mathbb{CP}^\infty$.

Applying the zeroth space functor to $bu \to gl_1(R)_Q$, we get an infinite loop map

(5) $\mathbb{Z} \times BU \to GL_1 R_Q$

which we then interpret as an invariant of stable bundles that carries Whitney sums to products, namely, a stable exponential characteristic class. Now we need to do the following simple calculation: we have the map (5); we know the composite $\mathbb{CP}^\infty \to \mathbb{Z} \times BU \to GL_1 R_Q$; we know that the map is multiplicative, and would like to calculate the effect of that map on $\pi_{2n}$. The answer turns out to be

(6) $K_\Phi(z) = \exp \left( \sum_{n=1}^{\infty} b_n \frac{z^n}{n!} \right)$,

where the constants $b_n$ are the same as the ones coming from the rightmost dotted arrow in (4).

If you’re trying to construct an orientation that realizes an old friend, such as the Witten genus, or the genus coming from the theorem of the cube, the formula (6) tells you which map you have to try to pick. In our case, the $b_n$’s were the coefficients in the log of the Weierstrass sigma function, which are Eisenstein series.

Let us take the example of $K$-theory and of the Todd genus $MU \to K$. The characteristic series is then given by

\[ K_\Phi(x) = \frac{x}{1 - e^{-x}}. \]

If we want to write it in the form (6), then we get

\[ d \log (K_\Phi(x)) = \frac{1}{x} - \frac{e^{-x}}{1 - e^{-x}} = \frac{1}{x} - \frac{1}{e^x - 1} = - \sum_{n \geq 0} B_n \frac{x^{n-1}}{n!}, \]

where $B_n$ are the Bernoulli numbers. By integrating and then exponentiating, we finally get $b_n = \frac{B_n}{n}$. If you do all that for the $\hat{A}$-genus [after replacing $bu$ by $bo$ in (4)], and once you decide what the canonical generators of $\pi_{4n}(bo)_Q$ are, you get $\frac{B_{2n}}{4n}$. These are the famous numbers from differential topology, whose denominators tell you the order of the image of $J$ [at odd primes], and whose numerators\footnote{The order of that group of exotic spheres is $2^{2n-2}(2^{2n-1} - 1)$ times the numerator of $\frac{B_{2n}}{4n}$.} tell you the order of group of smooth structure on $S^{4n-1}$ that bound parallelizable manifolds. All this is already a hint that there is going to be some connection to differential topology.

In geometric topology, the following groups play a central role: $O$ – the stable rotation group, $PL$ – the group of piecewise linear equivalences of the sphere, and $G = GL_1(S)$ – the self homotopy
equivalences of the sphere. All of them are infinite loop spaces, and there is a corresponding sequence of classifying spaces

\[ BO \to BPL \to BG = BGL_1(\mathbb{S}). \]

We also have associated spectra: \( \Sigma^{-1}bo \to \Sigma^{-1}bpl \to gl_1(\mathbb{S}). \) The corresponding homogeneous spaces form a fibration sequence

\[ (7) \quad PL/O \to G/O \to G/PL, \]

and they all have nice interpretations: the homotopy groups of \( G/O \) are called the structure sets of smooth structures on the sphere. The homotopy groups of \( G/PL \) are \( \mathbb{Z} \) in every fourth degree and zero everywhere else (at least at odd primes), and the map \( G/O \to G/PL \) sends an element of the structure set to its surgery obstruction, which is the signature. Finally, the trivializations of those obstructions are controlled by the space \( PL/O \), whose homotopy groups are the Kervaire-Milnor groups of smooth structures on the sphere.

Sullivan showed that the signature also induces a \( PL \) orientation \( MPL \to KO[\frac{1}{2}] \). That map is \( E_\infty \), and one of Sullivan’s results is that the corresponding map

\[ \Sigma^{-1}bpl \to gl_1(\mathbb{S}) \to g/pl \]

is a homotopy equivalence: \( g/pl \simeq gl_1 KO[\frac{1}{2}] \). If we now run the same story for oriented manifolds, we get

\[ (8) \quad \Sigma^{-1}bso \to gl_1(\mathbb{S}) \to g/so \]

and the dotted map \( C \to gl_1 KO[\frac{1}{2}] \) can be identified with the map \( g/so \to g/pl \) form the sequence (7). The upshot of that discussion is that the above map has something to do with geometric topology, and that its fiber is related to groups of exotic spheres.

So far, this was all about \( K \)-theory. But there is a conjectural analog for \( tmf \) that goes as follows. Let us now invert the prime 2 and consider \( tmf_0(2)^\wedge \), which is the value of the sheaf \( O^{\text{top}} \) on the moduli space of elliptic curves with a point of order two (which is étale over the moduli space of elliptic curves away from \( p = 2 \)). The same argument that refined the Witten genus will also refine the Ochanine genus, and will produce a map

\[ (9) \quad \Sigma^{-1}bso \to gl_1(\mathbb{S}) \to g/so \]

For \( p > 2 \), there is a pullback square

\[ F \to gl_1 tmf_0(2)^\wedge \to tmf_0(2)^\wedge_p \]

Here, \( F = \text{fib}(\ell) \), and the map \( \ell \) is called the topological logarithm as it maps the units of \( tmf_0(2) \) back to \( tmf_0(2) \). By construction, the Ochanine genus \( g/so \to gl_1 tmf_0(2)[\frac{1}{2}] \) maps to zero in
$tmf_0(2)_p$, and thus lifts to a map $g/so \rightarrow F$. The natural map from the diagram (8) to the diagram (9) therefore factors through the fiber of the topological logarithm.

In the $K$-theory story, the map $g/so \rightarrow gl_1(KO)[1/2]$ was intimately related to the group of exotic spheres. That story was built out of the signature. The Ochanine genus is supposed to be the signature of the loop space, so $\pi_*(F)$ should have something to do with exotic structures on the free loop space of spheres. Moreover, given an exotic structure on $S^n$ we clearly get one on $L^n$, and that’s what the map from (8) to (9) should be. Of course, this all completely conjectural... But this would be really neat prospect for $tmf$.

We now switch back to $tmf$ instead of $tmf_0(2)$, and denote by $F$ the fiber of the topological logarithm $gl_1(tmf)_p \rightarrow tmf_p$. The homotopy groups of $F$ are related to the spectrum of the Atkin operator, which has been intensely studied in number theory.

Let us look at $\pi_{23}(F)$. That group is equal to $\mathbb{Z}_p$ for $p = 691$, and is equal to $\mathbb{Z}_p \oplus \mathbb{Z}_p/(\tau(p) - 1)$ otherwise, where $\tau$ is the Ramanujan $\tau$-function given by

$$q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum \tau(n)q^n.$$ 

In particular, we see that $\pi_{23}(F)$ has a torsion group if and only if $\tau(p)$ is congruent to 1 mod $p$.

I have done a computer search up to $p \approx 10^9$, and the only solutions that I found are $p = 11, 23$, and 691. The number theorists believe that there should be infinitely many such primes, and that they should be log(log) distributed, but nobody really known how many solutions this equation has. Somehow, this is probably related to smooth structure on the loop space of the 23-sphere...

There are other strange things about the homotopy type of $F$. Aside from a $K$-theory summand that corresponds to classical geometric topology, the spectrum $F$ contains some $p$-adic suspensions of the image of $J$ spectrum. The first place one of them occurs (that you don’t understand) is at the prime 47. I tried to work out what $p$-adic suspension that was, computed nine 47-adic digits, but I couldn’t figure out what that number is. The number theorists to whom I showed it also couldn’t recognize it, which is strange, as one doesn’t expect homotopy theory to make up “random numbers”...

At present, we have to make the space $F$ at every prime separately. But if it really had to do with something awesome, like smooth structures on $L^n$, then it would probably come from a space that is just over $\mathbb{Z}$. If that was the case, then it would also be of finite type, and its homotopy groups would thus be finitely generated. As a consequence this would mean that there are only finitely many primes for which $\tau(p)$ is congruent to 1 mod $p$. Number theorists don’t really believe that this is possible... but who knows? I have no real reason to think that the spectra $F$ aren’t just reinvented at every prime.

Anyways, it would be really interesting to get to the bottom of that story, and blossom the relationship between number theory and smooth structures on spheres into new territory.