BOUSFIELD LOCALIZATION AND THE HASSE SQUARE

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1. Bousfield localization

The general idea of localization at a spectrum $E$ is to associate to any spectrum $X$ the “part of $X$ that $E$ can see”, denoted by $L_E X$. In particular, it is desirable that $L_E$ is a functor with the following equivalent properties:

$$E \wedge X \simeq * \implies L_E X \simeq *$$

If $X \to Y$ induces an equivalence $E \wedge X \to E \wedge Y$ then $L_E X \simeq L_E Y$.

**Definition 1.1.** A spectrum $X$ is called $E$-acyclic if $E \wedge X \simeq *$. It is called $E$-local if for each $E$-acyclic $T$, $[T,E] = 0$, where $[T,E]$ denotes the group of stable homotopy classes. A map of spectra $f: X \to Y$ is called an $E$-equivalence if $E \wedge f: E \wedge X \to E \wedge Y$ is a homotopy equivalence.

It is immediate that a spectrum $X$ is $E$-local iff for each $E$-equivalence $S \to T$, the induced map $[T,X] \to [S,X]$ is an isomorphism.

A spectrum $Y$ with a map $X \to Y$ is called an $E$-localization of $X$ if $Y$ is $E$-local and $X \to Y$ is an $E$-equivalence.

If a localization of $X$ exists, then it is unique up to homotopy and will be denoted by $X \xrightarrow{\eta} L_E X$.

Localizations of this kind were first studied by Adams [Ada74], but set-theoretic difficulties prevented him from actually constructing them. Bousfield found a way of overcoming these problems in the unstable category [Bou75]; for spectra, he showed in [Bou79] that localization functors exist for arbitrary $E$.

We start by collecting a couple of easy facts about localizations.

**Lemma 1.2.** Module spectra over a ring spectrum $E$ are $E$-local.

**Proof.** Since any map from a spectrum $W$ into an $E$-module spectrum $M$ can be factored through $E \wedge W$, it follows that all maps from an $E$-acyclic $W$ into $M$ are nullhomotopic. \qed

**Lemma 1.3.** If $v \in \pi_*(E)$ is an element of a ring spectrum $E$ (of an arbitrary but homogeneous degree), then $L_E \simeq L_{v^{-1}E \vee E/v}$, where $E/v$ denotes the cofiber of multiplication with $v$ and

$$v^{-1}E = \text{colim} \left( E \xrightarrow{v} E \xrightarrow{v} \cdots \right)$$

the mapping telescope.

**Proof.** It suffices to show that the class of $E$-acyclics agrees with the class of $(v^{-1}E \vee E/v)$-acyclics. Since $L_{v^{-1}E \vee E/v}$ is a module spectrum over $E$, $E$-acyclics are clearly $(v^{-1}E \vee E/v)$-acyclic; conversely, if $E/v \wedge W \simeq *$ then $v: E \wedge W \to E \wedge W$ is a homotopy equivalence, hence $E \wedge W \simeq v^{-1}E \wedge W$. Thus if also $v^{-1}E \wedge W \simeq *$, $W$ is $E$-acyclic. \qed

**Lemma 1.4.** Homotopy limits and retracts of $E$-local spectra are $E$-local.

**Proof.** The statement about retracts is obvious. For the statement about limits, first observe that a spectrum $X$ is $E$-local if and only if the mapping spectrum $\text{Map}(T,X)$ is contractible for all $E$-acyclic $T$. This is obvious because $\pi_k \text{Map}(T,X) = [\Sigma^k T, X]$, and if $T$ is $E$-acyclic then so are all its suspensions.

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Now if $F: I \to \{\text{spectra}\}$ is a diagram of $E$-local spectra, the claim follows from the equivalence
\[
\text{Map}(T, \text{holim } F) \simeq \text{holim } \text{Map}(T, F)
\]
\[\square\]

The following lemma characterizes $E$-localizations.

**Lemma 1.5.** The following are equivalent for a map of spectra $X \to Y$:

- $X \to Y$ is an $E$-localization;
- $X \to Y$ is the initial map into an $E$-local target;
- $X \to Y$ is the terminal map which is an $E$-equivalence.

**Proof.** Obvious from the axioms. \[\square\]

This characterization suggests two ways of constructing $X \to L_E X$:

- $L_E X = \text{holim}_{Y \text{-local}} X \to Y$ or
- $L_E X = \text{hocolim}_{E\text{-equivalence}} X \to Y$.

In both cases, these limits are not guaranteed to exist because the indexing categories are not small. This is more than a set-theoretic nuisance and requires a deeper study of the structure of the background categories.

I will first briefly discuss what can be done with approach $\circ$. The main construction will be closer to method $\bullet$.

**\circ Localizations as limits.** For a ring spectrum $E$, instead of indexing the homotopy limit over all $X \to Y$ with $Y E$-local, we could use the spaces in the Adams tower for $E$:
\[
X \to \text{Tot}^n \left( E^{\wedge(*)} \wedge X \right),
\]
which is a subdiagram because $E \wedge X$ is $E$-local for any $X$ by Lemma 1.2, and $E$-locality satisfies the 2-out-of-3 property for cofibration sequences of spectra. For this cosimplicial construction to make sense, the ring spectrum $E$ has to be associative in a strict sense (e.g. in the framework of [EKMM97]) or at least $A_\infty$ [BL10]), or one can restrict to cofacial spectra: A cofacial spectrum is a functor from $\Delta_f$ to spectra, where $\Delta_f$ is the subcategory of $\Delta$ with the same objects but only injective maps. In that case, Tot is just defined as the homotopy limit, and one can show that this agrees with the cosimplicial Tot if the cofacial spectrum is the underlying cofacial spectrum of a cosimplicial spectrum. Note that in this approach, no multiplication on $E$ is needed whatsoever – this works with any coaugmented spectrum $S \to E$.

If we are lucky, $X \to X_E \overset{\text{def}}{=} \text{Tot}(E^{\wedge(*)} \wedge X)$ is an $E$-localization. This is not always the case – $X \to X_E$ sometimes fails to be an $E$-equivalence. Whether or not $L_E X \simeq X_E$, the latter is what the $E$-based Adams-Novikov spectral sequence converges to and thus is of independent interest. If $L_E X$ can be built from $E$-module spectra by a finite sequence of cofiber extensions and retracts, then $L_E X \simeq X_E$ [Bou79, Thm 6.10] (such spectra are called $E$-prenilpotent). For some spectra $E$, every $X$ is $E$-prenilpotent; these spectra have the characterizing property that their Adams spectral sequence has a common horizontal vanishing line at $E_\infty$ and a horizontal stabilization line at every $E_r$ for every finite CW-spectrum [Bou79, Thm 6.12]. A necessary condition for this is that $E$ is smashing, i.e., that $L_E X = X \wedge L_E S^0$ for every spectrum $X$.

**\bullet Localizations as colimits.** Bousfield’s approach to constructing localizations uses colimits.

The basic idea for cutting down the size of the diagram the colimit is formed over is the following observation:

To check if $X$ is $E$-local, it is enough to show that for any $E$-equivalence $S \to T$ with $\#S$, $\#T < \kappa$ for some cardinal $\kappa$ depending only on $E$, $[T, X] \overset{\simeq}{\to} [S, X]$.

At this point, it is not crucial what exactly we mean by $\#S$. For a construction of $L_E X$ that is functorial up to homotopy, it is enough to define $\#S$ to be the number of stable cells.
Given this observation, $L_E X$ can be constructed in a small-object-argument-like fashion by forming homotopy pushouts

$$\prod_{S \to T \text{ E-equ.}} S \longrightarrow X \downarrow \quad \prod_{S \to T \text{ E-equ.}} T \longrightarrow X_{(1)}$$

and iterating this transfinitely (using colimits at limit ordinals). When the cardinal $\kappa$ is reached, $X_{(\kappa)}$ is $E$-local because it satisfies the lifting condition for “small” $S \to T$.

**Theorem 1.6.** The category of spectra has a model structure with

- cofibrations the usual cofibrations of spectra, i.e. levelwise cofibrations $A_n \to B_n$ such that $S^1 \wedge B_n \cup S^1 \wedge A_n \to B_{n+1}$ are also cofibrations;
- weak equivalences the (stable) $E$-equivalences;
- fibrations given by the lifting property

The fibrant objects in this model structure are the $E$-local $\Omega$-spectra.

Here are some explicit examples of localization functors.

**Example 1.7.**
1. $E = S^0$. In this case, $L_E$ is the functor that replaces a spectrum by an equivalent $\Omega$-spectrum.
2. $E = M(\mathbb{Z}(p)) =$ Moore spectrum. In this case $L_E X \simeq X_{(p)}$ is the Bousfield $p$-localization. This is an example of a smashing localization, i.e. $L_E X \simeq X \wedge L_E S^0$, which in this case is also the same as $X \wedge E$.
3. $E = M(\mathbb{Z}/p)$. For connective $X$, $L_E X \simeq X^p$ is the $p$-completion functor

$$X^p = \text{holim} \{ \cdots \to X \wedge M(\mathbb{Z}/p^2) \to X \wedge M(\mathbb{Z}/p) \}.$$ 

We write $X \overset{\eta_p}{\longrightarrow} L_p X$ for this localization $X \overset{\eta_p}{\longrightarrow} L_E X$.
4. $E = M(\mathbb{Q}) = H\mathbb{Q}$. As in (2), $L_E X = X \wedge L_E S^0 = X \wedge H\mathbb{Q}$ is smashing; it is the rationalization of $X$. As in the previous case, we write $X \overset{\eta_\mathbb{Q}}{\longrightarrow} L_Q X$ for this localization $X \overset{\eta_\mathbb{Q}}{\longrightarrow} L_E X$.

**2. The Sullivan arithmetic square**

The arithmetic square is a homotopy cartesian square that allows one to reconstruct a space if, roughly, all of its mod-$p$-localizations and its rationalization are known. For the case of nilpotent spaces, which is similar to spectra, this was first observed by Sullivan [Sul05].

**Lemma 2.1.** For any spectrum $X$, the following diagram is a homotopy pullback square:

$$\begin{array}{ccc}
X & \overset{\prod \eta_p}{\longrightarrow} & \prod_p L_p X \\
\downarrow_{\eta_\mathbb{Q}} & & \downarrow_{\eta_\mathbb{Q}} \\
L_Q X & \overset{L_Q(\prod \eta_p)}{\longrightarrow} & L_Q \left( \prod_p L_p X \right)
\end{array}$$

This is a special case of
Proposition 2.2. Let \( E, F, X \) be spectra with \( E_*(L_F X) = 0 \). Then there is a homotopy pullback square

\[
\begin{array}{ccc}
L_{E \vee F}X & \xrightarrow{\eta_E} & L_E X \\
\eta_F & & \downarrow \eta_F \\
L_F X & \xrightarrow{L_F(\eta_E)} & L_F L_E X
\end{array}
\]

In the case of Prop. 2.1, \( E = \bigvee_p M(\mathbb{Z}/p), F = H\mathbb{Q} \). To see that \( L_E = \prod_p L_p \), we have to show that there are no nontrivial homotopy classes from an \( E \)-acyclic space to a spectrum of the form \( \prod_p L_p X \), which is immediate, and that

\[
M(\mathbb{Z}/p)_*(X) \xrightarrow{\cong} M(\mathbb{Z}/p)_* \left( \prod_i L_i X \right)
\]

is an isomorphism for all \( p \). The latter holds because smashing with \( M(\mathbb{Z}/p) \) commutes with products since \( M(\mathbb{Z}/p) \) is a finite (two-cell) spectrum (use Spanier-Whitehead duality).

Furthermore, the condition \( E_*(L_F X) = E_*(H\mathbb{Q} \wedge X) = 0 \) is satisfied because \( H_*(M(\mathbb{Z}/p); \mathbb{Q}) = 0 \).

Proof of the proposition. Note that the map denoted \( \eta_E \) in the diagram is the unique factorization of \( \eta_E: X \to L_E X \) through \( L_{E \vee F} X \), which exists because \( X \to L_{E \vee F} X \) is an \( E \)-equivalence. The same holds for \( \eta_F \), and furthermore, these maps are \( E \)- and \( F \)-equivalences, respectively. Now let \( P \) be the pullback. We need to see that (1) \( P \) is \((E \vee F)\)-local and (2) the induced map \( X \to P \) is an \( E \)- and an \( F \)-equivalence. For (1), take a spectrum \( T \) with \( E_1 T = F_1 T = 0 \). Then in the Mayer-Vietoris sequence for the pullback,

\[
\cdots \to [T, P] \to [T, L_E X] \oplus [T, L_F X] \to [T, L_F L_E X] \to \cdots,
\]

the two terms on the right are zero, hence so is \([T, P]\).

For (2), observe that \( P \to L_E X \) is an \( F \)-equivalence because it is the pullback of \( \eta_F \) on \( L_E X \), and since \( X \to L_F X \) is also an \( F \)-equivalence, so is \( X \to P \). The same argument works for \( P \to L_E X \) except that here, the bottom map is an \( E \)-equivalence for the trivial reason that both terms are \( E \)-acyclic by the assumption. \( \square \)

3. Morava \( K \)-theories and related ring spectra

Given a complex oriented even ring spectrum \( E \) and an element \( v \in \pi_* E \), we would like to construct a new complex oriented ring spectrum \( E/v \) such that \( \pi_*(E/v) = (\pi_* E)/(v) \). This is clearly not always possible. The machinery of commutative \( S \)-algebras of [EKMM97] (or any other construction of a symmetric monoidal category of spectra, such as symmetric spectra) allows us to make this work in many cases where more classical homotopy theory has to rely on ad-hoc constructions (such as the Baas-Sullivan theory of bordism of manifolds with singularities).

In this section, let \( E \) be a complex oriented even commutative \( S \)-algebra and \( A \) an \( E \)-module spectrum with a commutative ring structure in the homotopy category of \( E \)-modules, and which is also a complex oriented even ring spectrum. Let us call this an \( E \)-even ring spectrum. A commutative \( E \)-algebra would of course be fine, but we need the greater generality.

Theorem 3.1 ([EKMM97, Chapter V]). For any \( v \in \pi_* E \), \( v^{-1} A \) carries the structure of an \( E \)-even ring spectrum. Furthermore, if \( v \) is a non-zero divisor then \( A/v \) is also an \( E \)-even ring spectrum.

Even if \( A \) is a commutative \( E \)-algebra (for example, \( A = E \)), the resulting spectrum is usually not a commutative \( S \)-algebra.

Of course, this construction can be iterated to give

Corollary 3.2. Given a graded ideal \( I \triangleleft \pi_* E \) generated by a regular sequence and a graded multiplicative set \( S \subset \pi_* E \), one can construct an \( E \)-even ring spectrum \( S^{-1} A/I \) with \( \pi_* S^{-1} A/I = S^{-1}(\pi_* A)/I \).
In particular, this can be done for $E = MU$. For example, $BP$ can be constructed in this way by taking $I = \ker(MU_* \to BP_*)$, which is generated by a regular sequence. It is currently not known whether $BP$ is a commutative $S$-algebra. However, the methods above allow us to construct all the customary $BP$-ring spectra by pulling regular sequence back to $E = MU_*$ and letting $A = BP$. For example,

\[
\begin{align*}
E(n) &= v_n^{-1}BP/(v_{n+1}, v_{n+2}, \ldots) \\
K(n) &= v_n^{-1}BP/(p, v_1, \ldots, v_{n-1}, v_{n+1}, \ldots) \\
P(n) &= BP/(p, v_1, \ldots, v_{n-1}) \\
B(n) &= v_n^{-1}BP/(p, v_1, \ldots, v_{n-1}).
\end{align*}
\]

Any $MU$-even ring spectrum $A$ gives rise to a Hopf algebroid $(A_*, A_+)$ and an Adams-Novikov spectral sequence

\[E^2_{**} = \text{Cotor}_{A_*, A_+}(A_*, A_+X) \Rightarrow \pi_+X_A.\]

If $M_A$ denotes the stack associated to the Hopf algebroid $(A_*, A_+)$ and $F_X$ the graded sheaf associated with the comodule $A_+X$, this $E^2$-term can be expressed as

\[E^2_{**} = H^{**}(M_A, F_X),\]

which is the cohomology of the stack $M_A$ with coefficients in the sheaf $F_X$.

In particular, if $f: A \to B$ is a morphism of $MU$-even ring spectra, we get a morphism of spectral sequences, and if $f$ induces an equivalence of the associated stacks, then $f$ induces an isomorphism of spectral sequences from the $E^2$-term on. In particular, in this case, $X_A \simeq X_B^\wedge$ if we can assure that the spectral sequences converge strongly. Note that we do not need an inverse map $B \to A$.

**Theorem 3.3.** If $f: A \to B$ is a morphism of $MU$-even ring spectra inducing an equivalence of associated stacks, then $L_A \simeq L_B$.

**Proof.** The argument outline above gives an almost-proof of this fact, but it puts us at the mercy of the convergence of the Adams-Novikov spectral sequences to the localizations $L_A X$ and $L_B X$. We give an argument that doesn’t require such additional assumptions. Note that it is sufficient to show that $A_+X = 0$ if and only if $B_+X = 0$. Assume $A_+X = 0$. Then the $A$-based Adams-Novikov spectral sequence is 0 from $E^1$ on, thus the $B$-based Adams-Novikov spectral sequence is also trivial from $E^2$ on. This time, the spectral sequence converges strongly because it is conditionally convergent in the sense of Boardman [Boa99], which implies strong convergence if the derived $E_\infty$-term is 0 – but this is automatic since the $E_r$-terms are all trivial for $r \geq 0$. Thus $X_B^\wedge$ is contractible.

Now the Hurewicz map $X \to B \wedge X$ factors as $X \to L_B X \to X_B^\wedge \to B \wedge X$ by the universal property of the localization, since $X_B^\wedge$ is $B$-local. Thus $X \to B \wedge X$ is trivial. Using the ring spectrum structure on $B$, we see that $B \wedge X \to B \wedge B \wedge X$ is $B \wedge X$, which is the identity, is also trivial, so $B \wedge X \simeq *$. \hfill \Box

In particular, this applies to the following cases:

**Theorem 3.4.** We have

\[L_{B(n)} \simeq L_{K(n)}.\]

Let $I_n = (p, v_1, \ldots, v_{n-1}) \subset BP_*$ and $E(k, n) = E(n)/I_k$ for $0 \leq k \leq n \leq \infty$. Then

\[L_{v_k^{-1}E(k, n)} \simeq L_{K(k)}.\]

**Proof.** The first part is due to Ravenel [Rav84] and Johnson-Wilson [JW75], but they give a different proof without the Adams-Novikov spectral sequence.

To apply Theorem 3.3, it is useful to extend the ground ring of the homology theories in question from $\mathbb{F}_p$ to $\mathbb{F}_{p^n}$, which does not change their localization functors. The Hopf algebroids for $B(n)\otimes \mathbb{F}_{p^n}$ and $K(n)\otimes \mathbb{F}_{p^n}$ both classify formal groups of height $n$. By Lazard’s theorem, there is only one such group over $\mathbb{F}_{p^n}$ up to isomorphism, which shows that the quotient map $B(n)\otimes \mathbb{F}_{p^n} \to K(n)\otimes \mathbb{F}_{p^n}$ induces an isomorphism of Hopf algebroids.
The second part works similarly by considering the maps of Hopf algebroids induced from 
\[ v_k^{-1}E(k, n) \leftarrow B(k)/(v_{n+1}, v_{n+2}, \ldots) \to K(k) \]
which again all represent the stack of formal groups of height \( k \).

\[ \square \]

**Theorem 3.5.** We have that 
\[ L_{E(n)} \simeq L_{K(0)\vee K(1)\vee \cdots \vee K(n)} \simeq L_{v_k^{-1}BP}. \]

**Proof.** With the notation of Theorem 3.4, since \( E(n, n) = K(n) \) and \( E(0, n) = E(n) \), it suffices to show that 
\[ L_{E(k, n)} \simeq L_{K(k)\vee E(k+1, n)}. \]

By Lemma 1.3, \( L_{E(k, n)} \simeq L_{v_k^{-1}E(k, n)\vee E(k+1, n)} \). By Theorem 3.4, \( L_{v_k^{-1}E(k, n)} \simeq L_{K(k)} \), and the result follows by induction.

The second equivalence can be proved by a similar argument, not needed here, and left to the reader. \( \square \)

**Theorem 3.6.** There is a homotopy pullback square

\[
\begin{array}{c}
L_{K(1)\vee K(2)}X \quad \eta_{K(2)} \quad L_{K(2)}X \\
\eta_{K(1)} \downarrow \quad \downarrow \eta_{K(1)} \\
L_{K(1)}X \quad L_{K(1)}(\eta_{K(2)}) \quad L_{K(1)}L_{K(2)}X
\end{array}
\]

**Proof.** This is an application of Prop. 2.2. We need to see that \( K(2)_*(L_{K(1)}X) = 0 \) for any \( X \).

To see this, let \( \alpha : \Sigma^k M(\mathbb{Z}/p) \to M(\mathbb{Z}/p) \) be the Adams map, which induces multiplication with a power of \( v_1 \) in \( K(1) \) and is trivial in \( K(2) \). Here \( k = 2p - 2 \) for odd \( p \) and \( k = 8 \) for \( p = 2 \).

Let \( X \) be \( K(1) \)-local. Then so is \( X \wedge M(\mathbb{Z}/p) \), and since \( \Sigma^k X \wedge M(\mathbb{Z}/p) \to X \wedge M(\mathbb{Z}/p) \) is a \( K(1) \)-isomorphism, it is a homotopy equivalence. On the other hand, \( \alpha_* : K(2)_*(\Sigma^k X \wedge M(\mathbb{Z}/p)) \to K(2)_*(X \wedge M(\mathbb{Z}/p)) \) is trivial, thus \( K(2)_*(X \wedge M(\mathbb{Z}/p)) = 0 \). By the Künneth isomorphism, \( K(2)_*(X) = 0 \). \( \square \)

The same result holds true for any \( K(m) \) and \( K(n) \) with \( m < n \); \( M(\mathbb{Z}/p) \) and \( \alpha \) then have to be replaced by a type-\( m \) complex and its \( v_m \)-self map in the argument. We briefly recall some basic facts around the periodicity theorem.

**Definition 3.7.** A finite \( p \)-local CW-spectrum \( X \) has type \( n \) if \( K(n)_*(X) \neq 0 \) but \( K(k)_*(X) = 0 \) for \( k < n \). For example, the sphere has type 0, the Moore spectrum \( M(\mathbb{Z}/p) \) has type 1, and the cofiber of the Adams map has type 2.

**Theorem 3.8** ([DHS88, HS98]). Every type-\( n \) spectrum \( X \) admits a \( v_n \)-self map, i.e. a map \( f : \Sigma^n X \to X \) which induces multiplication by a power of \( v_n \) in \( K(n)_*(X) \).

The periodicity theorem implies that there exist type-\( n \) complexes for every \( n \in \mathbb{N} \). They can be constructed iteratively, starting with the sphere, by taking cofibers of \( v_n \)-self maps. Thus, there exist multi-indices \( I = (i_0, \ldots, i_{n-1}) \) and spectra \( S^{i_0}/(v^{i_1}) \) such that \( BP_*(S^{i_0}/(v^{i_1})) = BP_*/(v^{i_1}) \), where \( (v^{i_1}) = (p^{i_0}, v_{i_1}, \ldots, v_{i_{n-1}}) \). These are sometimes called generalized Moore spectra. It is an open question what the minimal values of \( I \) are (they certainly depend on the prime.)

4. The Hasse square

In this section, we will study algebraic interpretations of \( K(n) \)-localization in terms of formal groups and elliptic curves.
Proposition 4.1. Let \( E \) be a complex oriented ring spectrum over \( S_0(p) \) and define
\[
E' = \lim_{i \in \mathbb{N}^n} v_n^{-1} E/(p^{k_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}}).
\]
Here \( v_i \) are the images of the classes in BP of the same name under the orientation \( BP \to E \). Then \( L_{K(n)} E \simeq E' \).

Proof. We also denote by \( I_n \subset E_+ \) the image of the ideal of the same name in \( BP \). As \( v_n^{-1} E/I_n \) is a \( B(n) \)-module spectrum, it is \( B(n) \)-local by Lemma 1.2, thus by Theorem 3.4 also \( K(n) \)-local. Each spectrum \( v_n^{-1} E/(v^j) \) (using multi-index notation) is constructed from \( v_n^{-1} E/I \) by a finite number of cofibration sequences, thus it is also \( K(n) \)-local. Since homotopy limits of local spectra are again local (Lemma 1.4), \( E' \) is \( K(n) \)-local, and it remains to show that \( K(n)_*(E) \cong K(n)_*(E') \).

The coefficient rings of the Morava \( K \)-theories \( K(n) \) are graded fields, hence they have Künneth isomorphisms. Thus it suffices to show that \( E \wedge X \to E' \wedge X \) is a \( K(n) \)-equivalence for some \( X \) with nontrivial \( K(n)_*(X) \). Choose \( X = S^0/(v^j) \) to be a generalized Moore spectrum of type \( n \), for some multi-index \( J \). Then
\[
E' \wedge X \simeq \lim_{i \in \mathbb{N}^n} (v_n^{-1} E/(v^j) \wedge S^0/(v^j)) \simeq v_n^{-1} E/(v^j).
\]
Thus \( K(n)_*(E \wedge X) = K(n)_*(v_n^{-1} E/(v^j)) = K(n)_*(E' \wedge X) \).

Now we will specialize to an elliptic spectrum \( E \) defined over the ring \( E_0 \) with associated elliptic curve \( C_E \) over Spec \( E_0 \). Proposition 4.1 in particular tells us that
\[
\pi_0 L_{K(1)} E \cong \lim_{i} v_1^{-1} E_0/(p^i),
\]
which is the ring of functions on \( \text{Spf}(E_0)^{\text{ord}} \), the ordinary locus of the formal completion of Spec \( E_0 \) at \( p \), i.e. the sub-formal scheme over which \( C_E \) is ordinary. In particular, if \( E_0 \) is an \( \mathbb{F}_p \)-algebra, \( \pi_0 L_{K(1)} E \cong v_1^{-1} E_0 \) is just the (non-formal) ordinary locus of \( E_0 \). Similarly,
\[
\pi_0 L_{K(2)} E \cong \lim_{i_0, i_1} v_2^{-1} E_0/(p^{i_0}, v_1^{i_1}) = \lim_{i_0, i_1} E_0/(p^{i_0}, v_1^{i_1})
\]
is the ring of functions on the formal completion of Spec \( E_0 \) at the supersingular locus at \( p \). The last equality holds because any elliptic curve has height either 1 or 2 over \( \mathbb{F}_p \), thus \( v_2 \) is a unit in \( E_0/(p, v_1) \) and hence in \( E_0/(p^{i_0}, v_1^{i_1}) \).

Lemma 4.2. Any \( p \)-local elliptic spectrum \( E \) is \( E(2) \)-local.

Proof. We need to show that for any \( W \) with \( E(2), W = 0 \), we have that \( E, W = 0 \). By Theorems 3.4 and 3.5, this is equivalent to \( B(i) = 0 \) for \( 0 \leq i \leq 2 \). That is,
\[
\begin{align*}
p^{-1} BP \wedge W & \simeq * \\
v_1^{-1} BP/p \wedge W & \simeq * \\
v_2^{-1} BP/(p, v_1) \wedge W & \simeq *.
\end{align*}
\]
Now since \( E \) is a \( BP \)-ring spectrum, the same equalities hold with \( BP \) replaced by \( E \). It follows from Lemma 1.3 that
\[
\begin{align*}
E/(p, v_1) \wedge W & \simeq v_2^{-1} E/(p, v_1) \wedge W \simeq * \quad \text{and} \quad v_1^{-1} E/p \wedge W \simeq * \quad \Rightarrow \quad E/p \wedge W \simeq * \\
E/p \wedge W & \simeq * \quad \text{and} \quad p^{-1} E \wedge W \simeq * \quad \Rightarrow \quad E \wedge W \simeq *.
\end{align*}
\]
\)

Corollary 4.3 (the “Hasse square”). For any elliptic spectrum \( E \), there is a pullback square
\[
\begin{array}{ccc}
E_0 & \to & L_{K(2)} E \\
\downarrow & & \downarrow \\
L_{K(1)} E & \to & L_{K(1)} L_{K(2)} E.
\end{array}
\]
Proof. It follows from Lemma 3.6 that the pullback is $L_{K(1)\vee K(2)}E$. Now consider the arithmetic square

$$
\begin{array}{ccc}
L_{K(0)\vee K(1)\vee K(2)}E & \rightarrow & L_{K(1)\vee K(2)}E \\
\downarrow & & \downarrow \\
L_{K(0)}E & \rightarrow & L_{K(0)\vee K(1)\vee K(2)}E.
\end{array}
$$

Since $L_pL_{K(0)}X = L_pL_0X = \ast$, applying the $p$-completion functor $L_p$, we see that top horizontal map

$$L_pE \simeq L_pL_{K(0)\vee K(1)\vee K(2)}E \rightarrow L_pL_{K(1)\vee K(2)}E \simeq L_{K(1)\vee K(2)}E$$

is an equivalence, hence the result. □

References


