The goal of the workshop is to provide an introduction to the Strominger-Yau-Zaslow conjecture in mirror symmetry, leading up to a (partial) account of the Gross-Siebert program (as announced in [GS03], and developed in a series of subsequent papers by the same authors) and aspects of recent works such as Gross-Pandharipande-Siebert [GPS] or Gross-Hacking-Keel [GHK]. The principal goal would be to motivate and present Gross and Siebert’s algebro-geometric interpretation of the SYZ conjecture, and the ingredients thereof; as well as some primer lectures on SYZ, topics will include affine manifolds, aspects of log geometry and tropical geometry, theta functions and some of the work of Kontsevich and Soibelman [KS].

A number of surveys and expository works exist on these topics, which will be used as the principal learning sources, at least for the first part of the workshop; aside from the comprehensive Mirror Symmetry monograph [Clay03] and the more recent Dirichlet Branes and Mirror Symmetry monograph [Clay09], the recent survey article by Mark Gross [CDM] and the expository article [GS08] should both be reasonably accessible to workshop participants. (Chap. 6 through 8 of [Clay09] should be of particular interest; notably, the viewpoint in Chap. 8 makes connections with theta functions, relating to [GS11b].)

References include:

Surveys and exposition:


“Landmark” papers (also background)

[SYZ] A. Strominger, S.-T. Yau, E. Zaslow, Mirror symmetry is T-duality

[PZ] A. Polishchuk and E. Zaslow, Categorical mirror symmetry for the elliptic curve


[GS03] M. Gross and B. Siebert, Affine manifolds, log structures and mirror symmetry, 2003


**Day 1.**

**Talk 1.** Overview of mirror symmetry. (Gross) A historical introduction to mirror symmetry, putting in context the Gross-Siebert program.

**Talk 2.** The SYZ conjecture. A brief description of the original conjecture [SYZ]. Given a special Lagrangian fibration $f : X \to B$, we explore the structures appearing on the base $B$, as explained in [Clay09], Chapter 6 and [G08], §2. These structures include two affine structures, a multi-valued convex function determining a metric, and a notion of Legendre transform.

**Talk 3.** Semi-flat mirror symmetry. Starting with a tropical affine manifold $B$, one obtains a complex manifold $X(B)$ and a symplectic manifold $\hat{X}(B)$. [Clay09], Chapter 6 explores in details some of the data arising on $X(B)$ and $\hat{X}(B)$ from data on $B$.

**Talk 4.** Topological mirror symmetry. Calabi-Yau hypersurfaces in toric varieties give rise to integral affine manifolds with singularities (§4 of [G08]), with $B_0 \subseteq B$ a subset carrying an integral affine structure and with $\Delta := B \setminus B_0$ codimension $\geq 2$ in $B$. Then $X(B_0)$ or $\hat{X}(B_0)$ exist. In three dimensions, in nice cases (the “simple case”), these can be
compactified to torus fibrations $f : X(B) \to B$ and $\tilde{f} : \tilde{X}(B) \to B$. This was carried out in [G99], but is explained in sufficient detail in [Clay09], §6.4. This is also a good point to talk about how the Leray spectral sequence for $f$ and $\tilde{f}$ explain the interchange of Hodge numbers in three dimensions, see §6.4.4 of [Clay09] or [G08], §1.

**Day 2**

**Motivational 15 minute talk (Gross):** Outline of what the Gross-Siebert program hopes to accomplish.

**Talk 5.** Mirror symmetry is largely concerned with the behaviour of degenerating families of Calabi-Yau manifolds (or more generally manifolds with effective anti-canonical class). [G08], §7 explains the type of degeneration considered by the Gross-Siebert program, toric degenerations. A toric degeneration gives rise to the dual intersection complex, which is an integral affine manifold with singularities, and a polarized toric degeneration gives rise to the intersection complex. The two are related via a discrete Legendre transform, giving a discrete analogue of the structures appearing in the SYZ picture.

**Talk 6.** The key point of the Gross-Siebert program was the hope that one could go backward, starting with an integral affine manifold $B$ with singularities and some additional data and producing a toric degeneration whose (dual) intersection complex is $B$. This was accomplished in [GS07]. The latter paper is fairly inpenetrable, so [GS08], [CDM], §10, 11 and [TGMS], Chapter 6 should be made use of. We should stick to the two-dimensional case, which is notationally much simpler, and the argument in that class follows [KS] fairly closely. In the this talk, the Mumford degeneration can be covered in detail: see [GS08], §1 and [TGMS], §6.2.1. This covers the case when $B$ is a lattice polytope.

**Talk 7.** The case for $B$ general without singularities can now be covered ([TGMS], §6.2.2). This requires developing an understanding of various local systems living on $B$. This also makes connection to the construction of the Tate elliptic curve in the case when $B = \mathbf{R}/d\mathbf{Z}$ is a circle. This case is discussed in [TGMS], §6.2.2 as well as [Clay09], §8.4. If time permits, there can be some discussion as to what goes wrong in the case of singularities: see [GS08], §2.3 and [TGMS], §6.2.3.

**Talk 8.** The case for $B$ with singularities, without scattering. Cover the examples of [GS08], §3. The crucial point is to explain how modifying gluing maps via automorphisms attached to walls fixes the problems caused by monodromy. These corrections can also be explained in terms of Maslov index zero disks in the Fano context; this is a good point at which to make contact with Auroux’ work [A07].

**Day 3.**
**Talk 9.** Discussion of the scattering process. Start with the examples of [GS08], §4.1, 4.2. Discuss the Kontsevich-Soibelman lemma in dimension two: see [GPS], Theorem 1.4 or [TGMS] §6.3.1. Finish the outline of the algorithm for producing smoothings in dimension two, following the outline in [CDM], §10 and all details covered in [TGMS], Chapter 6.

**Talk 10.** The enumerative interpretation of the Kontsevich-Soibelman lemma as given in [GPS]. The survey [GP09], as well as [CDM] §11 or [TGMS], §6.3.2 can be consulted. This gives evidence that the algorithm for producing smoothings morally is correcting the complex structure using Maslov index zero disks on the mirror side, again fitting with the philosophy of [A07].

**Talk 11.** Introduction to homological mirror symmetry [K]. Many sources can be used to prepare this material. Explain enough to be able to give the motivation for theta functions, as outlined in §1 of [GS11b].

**Talk 12.** Talk about the idea of how tropical trees can approximate holomorphic disks contributing to Floer homology between sections. See §8.4 of [Clay09] (WARNING: There is a mistake in this section: if input points coincide, the moduli space of tropical trees is not the correct dimension and some perturbation methods are necessary. This doesn’t arise when computing $m_2$, so this is sufficient for our purposes) and §2 of [GS11b].

**Day 4.**

**Talk 13.** Explain the notion of jagged path ([GS11b], §3.2) and how it gives rise to a construction of theta functions, motivated by the tropical interpretation of Floer homology. Cover in detail the examples of this section, reprising the discussion of these examples in Talk 8.

**Talk 14.** Begin discussing [GHK]. A revised, shorter and more focused version of this paper is currently available at http://www.math.ucsd.edu/~mgross/GHK23.pdf

§5 of [GS11b] gives a little bit of exposition of the ideas in this paper. Begin by explaining how a pair $(Y, D)$ of rational surface $Y$ and $D \in |-K_Y|$ a cycle of rational curves gives rise to an integral affine manifold with singularities. Review scattering diagrams in this context and how they give rise to deformations of the punctured $n$-vertex. Describe the variant of jagged paths called broken lines and how consistency gives the existence of theta functions. Explain how these are used to extend the deformations of punctured vertex to deformations of the vertex. These ideas are largely variations of ideas seen earlier in the week.

**Talk 15.** Finish the discussion of [GHK]. Explain the canonical scattering diagram,
making contact with [GPS] and the Kontsevich-Soibelman lemma. This is §3 of the new version of [GHK]. The basic idea is that using the existence of a toric model and the main result of [GPS], one converts the scattering diagram on the affine manifold associated to \((Y, D)\) to one in \(\mathbb{R}^2\) which is generated by the Kontsevich-Soibelman process. Carl, Pumperla and Siebert then showed such scattering diagrams are consistent.

**Talk 16.** So far, we have explicitly avoided mentioning log geometry. Nevertheless, log geometry was present at the beginning of the Gross-Siebert program, but many of the points discussed in earlier lectures, while depending implicitly on log geometry, can be discussed without mentioning log geometry explicitly.

It is now time to introduce log geometry into the discussion. The first two lectures should give a very gentle introduction to log geometry, following Chapter 3 of [TGMS].

**Day 5.**

**Talk 17.** This should continue with the introduction to log geometry. In particular, log differentials and some log deformation theory should be covered, see [TGMS], §§3.3 and 3.4.

**Talk 18.** We now turn to specific applications of log geometry for mirror symmetry. In the Gross-Siebert program, log structures appear on both the A- and B-model sides of mirror symmetry. On the B-model side, in [GS03b], we considered the degenerate fibre of a toric degeneration along with a natural log structure on it. This is called a log Calabi-Yau space. In particular, in nice cases, [GS03b] constructs moduli spaces of log Calabi-Yau spaces which have the expected dimension. In [GS07b], direct calculations of log de Rham cohomology are carried out for log Calabi-Yau spaces, demonstrating the interchange of Hodge numbers. This situation is surveyed in §8 and §9 of [G08]. It would be a good idea to outline this material without going into too many technical details.

**Talk 19.** On the A-model side, the expectation is that Gromov-Witten invariants can be computed on log Calabi-Yau spaces using logarithmic Gromov-Witten invariants. One replaces ordinary stable maps in Gromov-Witten theory with stable log maps from a log curve to a logarithmic target space. Frequently it is easier to carry out Gromov-Witten calculations on the degenerate space. A model for how this is done is [NS04]. The authors prove a version of Mikhalkin’s formula for counts of holomorphic curves in toric varieties in all dimensions using the idea of degenerating the toric variety to a union of toric varieties (a toric degeneration) suitably chosen to fit with tropical geometry. These ideas should be surveyed. [TGMS], Chapter 4, covers the proof of [NS04] in the two-dimensional case, which is combinatorially simpler.
Talk 20. This leads naturally to the notion of logarithmic Gromov-Witten invariants, as developed in [GS11c]. The theory should be outlined, with special emphasis on the relationship with tropical geometry.