Problem 8. Let \((A_i : i \geq 1)\) be a sequence of countable sets. Show that 
\[
\bigcup_{i \in \mathbb{N}\setminus\{0\}} A_i
\]
is also countable.

Proof.
Problem 9. Let \( f : (0, 1) \to \mathbb{R} \) be given and suppose that \( x \mapsto f(x) \) has a limit as \( x \to c \) for some \( c \in (0, 1) \). Show that the limit is unique – that is, suppose that \( a, b \) are limits of \( f \) as \( x \to c \) and show that \( a = b \).

Proof.

Problem 10. Define \( f : \mathbb{R} \to \mathbb{R} \) by the conditions

\[
  f(x) = 1 \text{ for } x \in \mathbb{Q}
\]

and

\[
  f(x) = -1 \text{ for } x \in \mathbb{R} \setminus \mathbb{Q}.
\]

Show that, for every \( x_0 \in \mathbb{R} \), \( f(x) \) does not have a limit as \( x \to x_0 \). (In other words, given \( x_0 \in \mathbb{R} \) and \( \lambda \in \mathbb{R} \), show that the condition

\[
  \lim_{x \to x_0} f(x) = \lambda
\]

is not valid. You may use the fact that for any \( z \in \mathbb{R} \) and \( \epsilon > 0 \) there exists \( q \in \mathbb{Q} \) with \( |z - q| < \epsilon \).

Proof.
Problem 11. Let \( f : (0,1) \rightarrow (0,1) \) be a surjective function which is strictly increasing in the sense that for every \( x, y \in (0,1) \) with \( x < y \) we have \( f(x) < f(y) \). Show that \( f \) is continuous.

(Recall that \( f \) surjective (alternatively, onto) means that for every \( y \in (0,1) \) there exists \( x \in (0,1) \) with \( f(x) = y \)).

Proof.