Introduction to Proofs
IAP 2015

Solution to first in-class problem for day 2

Problem 2. Let $X$ and $Y$ be sets, and let $f : X \to Y$ be a given function. Show that for all $A, B \subseteq Y$,

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$$

and for all $U, V \subseteq X$,

$$f(U \cup V) = f(U) \cup f(V).$$

Hint: The result can be divided into four separate implications.

Proof. Let $A, B \subseteq Y$ be given. We first show that

$$f^{-1}(A \cap B) \subseteq f^{-1}(A) \cap f^{-1}(B).$$

For this, let $x \in f^{-1}(A \cap B)$ be given, so that $f(x) \in A \cap B$, and therefore both of the conditions $f(x) \in A$ and $f(x) \in B$ hold. Rewriting these conditions, we obtain $x \in f^{-1}(A)$ and $x \in f^{-1}(B)$, so that $x \in f^{-1}(A) \cap f^{-1}(B)$ as desired. Since $x \in f^{-1}(A \cap B)$ was arbitrary, this shows the desired implication.

We now prove

$$f^{-1}(A) \cap f^{-1}(B) \subseteq f^{-1}(A \cap B).$$

Let $x \in f^{-1}(A) \cap f^{-1}(B)$ be given. This implies that both $f(x) \in A$ and $f(x) \in B$ hold, so that we have $f(x) \in A \cap B$. Thus $x \in f^{-1}(A \cap B)$ as desired.

Let $U \subseteq X$ and $V \subseteq X$ be given. We first demonstrate the inclusion

$$f(U \cup V) \subseteq f(U) \cup f(V).$$

For this, let $x \in f(U \cup V)$ be given. We can then choose $y \in U \cup V$ such that $x = f(y)$. If $y$ belongs to the set $U$, then we immediately conclude $x \in f(U)$ (and, since $f(U) \subseteq f(U) \cup f(V)$ holds trivially, $x \in f(U) \cup f(V)$). On the other hand, if $y$ does not belong to $U$, $y \in U \cup V$ implies that $y$ belongs to $V$, and we therefore have $x \in f(V) \subseteq f(U) \cup f(V)$. Thus, in either case, we have $x \in f(U) \cup f(V)$, which implies the desired inclusion.

It remains to establish that

$$f(U) \cup f(V) \subseteq f(U \cup V).$$

Let $x \in f(U) \cup f(V)$ be given. Suppose first that $x \in f(U)$. It then follows that there exists $y \in U$ such that $x = f(y)$. Since $U \subseteq U \cup V$, we therefore have $y \in U \cup V$, and thus $x \in f(U \cup V)$. Alternatively, if $x \notin f(U)$, the condition $x \notin f(U) \cup f(V)$ implies $x \notin f(V)$, which gives the existence of some $y \in V \subseteq U \cup V$ with $x = f(y)$. This in turn implies $x \in f(U \cup V)$ as desired. Since $x \in f(U \cup V)$ holds in either case, the desired inclusion follows.

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