Homework 3*

1. In $\mathbb{R}^3$ with the standard metric, let $M$ be the manifold parametrized by $(x, y, z) = (r \cos(\theta), r \sin(\theta), \theta)$, and let $N$ be the manifold parametrized by $(x, y, z) = (\cosh(u) \cos(v), \cosh(u) \sin(v), u)$.
   ▷ a: Sketch $M$ and $N$.
   ▷ b: Calculate the restriction of $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ for $M$ and $N$, with respect to their coordinates $(r, \theta)$ and $(u, v)$.
   ▷ c: Show that the map $M \to N$ given by $\theta = v$ and $r = \sinh(u)$ is an isometry.
   ▷ d: Show that there is no isometry of $\mathbb{R}^3$ taking $M$ to $N$, even locally.

2a: Give $\mathbb{R}^N$ the standard metric. Let $M \subseteq \mathbb{R}^N$ be a submanifold, and assume that $\|\cdot\|$ is identically zero. Show that $M$ is an affine linear subspace.
   ▷ b: Suppose that $q \in M$ is a point where $\|\cdot\|$ is not identically zero. Show that there is a point $p \in \mathbb{R}^N \setminus M$ so that $q$ is a degenerate critical point of $d^2_p$.
   ▷ c: Let $M \subseteq \mathbb{R}^3$ be the helix curve, given by $(x, y, z) = (\cos(t), \sin(t), t)$. Find all points $p \in \mathbb{R}^3$ so that $d^2_p$ is not Morse on $M$.

3a: Let $f : M \to \mathbb{R}$ be a Morse function. Show that $\chi(M) = \sum_{p \in \text{crit}(f)} (-1)^{\text{ind}(p)}$.
   ▷ b: Let $\Sigma$ be a closed surface with $\chi(\Sigma) \leq 1$. Show that any Morse function on $\Sigma$ must have at least $|\chi(\Sigma)| + 2$ critical points.
   ▷ c: Let $M$ be a closed orientable 3-manifold, and let $f : M \to \mathbb{R}$ be a Morse function with four critical points, which has distinct critical values. Show that there is an embedded circle $S^1 \subseteq M$, so that $M \setminus S^1 \cong S^1 \times \mathbb{R}^2$. You may assume two facts without proof: the only boundaryless surface which is homotopy equivalent to a point is $\mathbb{R}^2$, and every orientation preserving diffeomorphism of $\mathbb{R}^2$ is isotopic to the identity. (Hint: If $c \in \mathbb{R}$ has two critical values below it and two above it, what is the topology of $f^{-1}((-\infty, c])$? Use cases.)
   ▷ d: Give an example of a closed 3-manifold $M$, so that any Morse function on $M$ has at least 6 critical points.

*Due November 13th