18.721 Assignment 9

This assignment is due Wednesday, May 11.

Let $Y$ be a projective curve of genus $g > 0$. The module $\Omega_Y$ of differentials is isomorphic to $\mathcal{O}(K)$ for some divisor $K$ of degree $2g - 2$, called a canonical divisor, which is determined up to linear equivalence. Because the canonical module $\mathcal{O}(K)$ has nonzero global sections, there is an effective canonical divisor $K$.

If $D$ is a divisor, the Serre dual of the module $\mathcal{O}(D)$ is $\mathcal{O}(K - D)$.

1. Suppose that $g = 2$.
   (a) Determine all possible dimensions of $H^q(Y, \mathcal{O}(D))$, when $D$ is an effective divisor of degree $n$.
   (b) Let $K$ be an effective canonical divisor. Then 1 is a global section of $\mathcal{O}(K)$, and there is also a nonconstant global section $x$. Prove that the pair of functions $(1, x)$ defines a morphism $Y \to \mathbb{P}^1$ that represents $Y$ as a double cover of the projective line.
   (c) Determine the number of branch points of this double covering.

2. Suppose that $g = 3$, an let $K$ be and effective canonical divisor.
   (a) Let $(1, x, y)$ be a basis for $H^0(Y, \mathcal{O}(K))$. Use Riemann-Roch for multiples of $K$ to show that $x, y$ satisfy a polynomial relation of degree at most 4.
   (b) Let $f$ be the morphism from $Y$ to $\mathbb{P}^2$ defined by the rational functions $(1, x, y)$. Show that the image $C$ of $f$ is a plane curve of degree at most 4, and that if its degree is 4, then $C$ is a smooth curve.