18.721 Assignment 7

This assignment is due Wednesday, May 16, the last day of class.

1. The projective line \( X = \mathbb{P}^1 \) with coordinates \( x_0, x_1 \) is covered by the two standard affine open sets \( U_0 = \text{Spec} \, R_0 \) and \( U_1 = \text{Spec} \, R_1 \), \( R_0 = \mathbb{C}[u] \) with \( u = x_1/x_0 \), and \( R_1 = \mathbb{C}[v] \) with \( v = x_0/x_1 = u^{-1} \). The intersection \( U_{01} \) is the spectrum of the Laurent polynomial ring \( R_{01} = \mathbb{C}[u,v] = \mathbb{C}[u,u^{-1}] \). The units of \( R_{01} \) are the monomials \( cu^k \), where \( k \) can be any integer.

   (a) Let \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) be an invertible \( R_{01} \)-matrix. Prove that there is an invertible \( A_0 \)-matrix \( Q \) and there is an invertible \( A_1 \)-matrix \( P \) such that \( Q^{-1}AP \) is diagonal.

   (b) Use part (a) to prove the Birkhoff-Grothendieck Theorem for torsion-free \( O_X \)-modules of rank 2: Such a module is isomorphic to a direct sum of twisting modules: \( O(m) \oplus O(n) \).

2. Let

\[
N = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \\ y_{31} & y_{32} \end{pmatrix}
\]

be a \( 3 \times 2 \) matrix whose entries are homogeneous polynomials of degree \( d \) in \( R = \mathbb{C}[x_0, x_1, x_2] \), and let \( M = (m_1, m_2, m_3) \) be the \( 1 \times 3 \) matrix of minors

\[
m_1 = y_{21}y_{32} - y_{22}y_{31}, \quad m_2 = -y_{11}y_{32} + y_{12}y_{31}, \quad m_3 = y_{11}y_{22} - y_{12}y_{21}.
\]

Let \( I \) be the ideal of \( R \) generated by the minors \( m_1, m_2, m_3 \).

(a) Prove that if \( I \) is the unit ideal of \( R \), the sequence

\[0 \leftarrow R \xleftarrow{M} R^3 \xleftarrow{N} R^2 \leftarrow 0\]

is exact.

(We’ve written the arrows from right to left here so that matrix multiplication by \( M \) on \( R^3 \) and \( N \) on \( R^2 \) are defined, when elements of \( R^3 \) and \( R^2 \) are represented as column vectors.)
(b) Let $X = \mathbb{P}^2$, and suppose that the locus $Y$ of zeros of $I$ in $X$ has dimension zero. Prove that the sequence

$$0 \leftarrow R/I \leftarrow R^M \leftarrow R^2 \leftarrow 0$$

is exact.

(c) The sequence in (b) corresponds to the following sequence, in which the terms $R$ are replaced by twisting modules:

$$0 \leftarrow \mathcal{O}_Y \leftarrow \mathcal{O}_X^M \leftarrow \mathcal{O}_X(-2d)^3 \leftarrow \mathcal{O}_X(-3d)^2 \leftarrow 0$$

Use this sequence to determine $h^0(Y, \mathcal{O}_Y)$. Check your work in some example in which $y_{ij}$ are homogeneous linear polynomials.

The background required for the next two problems is in Chapter 7 of the notes, which will be discussed in class next week. Let $Y$ be a projective curve of genus $g > 0$. The module $\Omega_Y$ of differentials is isomorphic to $\mathcal{O}(K)$ for some divisor $K$ of degree $2g - 2$, called a canonical divisor, which is determined up to linear equivalence. Because the canonical module $\mathcal{O}(K)$ has nonzero global sections, there is an effective canonical divisor $K$.

3. Suppose that $g = 2$.

(a) Determine the possible dimensions of $H^q(Y, \mathcal{O}(D))$, when $D$ is an effective divisor of some given degree $n$.

(b) Let $K$ be an effective canonical divisor. Then 1 is a global section of $\mathcal{O}(K)$, and there is also a nonconstant global section $x$. Prove that the pair of functions $(1, x)$ defines a morphism $Y \rightarrow \mathbb{P}^1$ that represents $Y$ as a double cover of the projective line.

4. Suppose that $g = 3$, an let $K$ be and effective canonical divisor.

(a) Let $(1, x, y)$ be a basis for $H^0(Y, \mathcal{O}(K))$. Use Riemann-Roch for multiples of $K$ to show that $x, y$ satisfy a polynomial relation of degree at most 4.

(b) Let $f$ be the morphism from $Y$ to $\mathbb{P}^2$ defined by the rational functions $(1, x, y)$. Show that the image $C$ of $f$ is a plane curve of degree at most 4, and that if its degree is 4, then $C$ is a smooth curve.