This assignment is due Friday, February 19

1. (a) Let $f$ a homogeneous polynomial in $x, y, z$, not divisible by $z$. Prove that $f$ is irreducible if and only if $f(x, y, 1)$ is irreducible.
(b) Prove that most nonhomogeneous polynomials in two or more variables are irreducible.

2. Let $f(x, y, z)$ and $g(x, y, z)$ be homogeneous polynomials of degrees $m$ and $n$, with no common factor, let $R$ be the polynomial ring $\mathbb{C}[x, y, z]$, and let $A = R/(f, g)$.
(a) Show that the sequence

$$0 \to R \xrightarrow{(-g, f)} R^2 \xrightarrow{(f, g)^t} R \to A \to 0$$

is exact.
(b) (algebraic version of Bézout’s Theorem) Because $f$ and $g$ are homogeneous, $A$ inherits a grading by degree, i.e., $A = A_0 \oplus A_1 \oplus \cdots$, where $A_n$ is the image in $A$ of the space $R_n$ consisting of the homogeneous polynomials of degree $n$ together with 0. Prove that $\dim A_k = mn$ for all sufficiently large $k$.

3. Let $p$ be a cusp of the curve $C$ defined by a homogeneous polynomial $f$. Prove that there is just one line $\ell$ through $p$ such that the restriction of $f$ to $\ell$ has as zero of order $> 2$, and that the order of zero for this line is precisely 3.

4. Exhibit an irreducible homogeneous polynomial $f(x, y, z)$ of degree 4 whose locus of zeros is a curve with three cusps.