2. Let \( f \) and \( g \) be irreducible homogeneous polynomials in \( x, y, z \). Prove that if the loci \( \{ f = 0 \} \) and \( \{ g = 0 \} \) are equal, then \( g = cf \).

We write \( f \) and \( g \) as polynomials in \( z \) whose coefficients are polynomials in \( x, y \), and we embed \( R = \mathbb{C}[x, y, z] \) into the ring \( F[z] \), where \( F = \mathbb{C}(x, y) \).

Suppose that \( g \) isn’t a constant multiple of \( f \). Then because these polynomials are irreducible, they have no common factor in \( R \). They can’t have a common factor in \( F[z] \) either. If \( h \) was a common factor in \( F[z] \), one could clear denominators to make \( h \) an element of \( R \), and replace \( h \) by an irreducible factor in \( R \) that involves \( z \). Then since \( h \) divides \( f \) in \( F[z] \), it divides in \( R \) too.

This being so, one can write \( pf + qg = 1 \), with \( p, q \) in \( F[z] \). Clearing denominators gives an equation in \( R \) of the form \( \hat{p}f + \hat{q}g = d(x, y) \). Then for any point \((x_0, y_0)\) such that \( d(x_0, y_0) \neq 0 \), \( f(x_0, y_0, z) \) and \( g(x_0, y_0, z) \) have no common zeros.

4. Prove that a plane cubic curve can have at most one singular point.

Suppose that the cubic curve \( C \) is singular. We choose coordinates so that the singular point is \( p = (0, 0, 1) \). Let the equation for \( C \) be \( f(x, y, z) = 0 \). Then \( f(p) = 0 \) because \( p \in C \), and \( f_x(p) = f_y(p) = 0 \) because \( p \) is a singular point. This implies that the coefficients of the monomials \( z^3, xz^2, yz^2 \) in \( f \) are zero. If there were another singular point, we could put it at \( q = (1, 0, 0) \), and the same reasoning would show that the coefficients of \( x^3, x^2y, x^2z \) in \( f \) are zero. Then \( f \) would be a combination of \( y^3, xy^2, y^2z, xyz \), and would be divisible by \( y \), contradicting irreducibility.

6. Let \( C \) be a smooth cubic curve in \( \mathbb{P}^2 \), and let \( p \) be a flex point of \( C \). Choose coordinates so that \( p \) is the point \((0, 1, 0)\) and the tangent line to \( C \) at \( p \) is the line \( \{ z = 0 \} \).

(a) Show that the coefficients of \( x^2y, xy^2, \) and \( y^3 \) in the defining polynomial \( f \) of \( C \) are zero.

(b) Show that with a suitable choice of coordinates, one can reduce the defining polynomial to the form \( f = y^2z + x^3 + axz^2 + bz^3 \), where \( x^3 + ax + b \) is a polynomial with distinct roots.

(c) Show that one of the coefficients \( a \) or \( b \) can be eliminated.

(b) With coordinates as indicated, the cubic polynomial has the form \( f(x, y, z) = y^2z + \ell(x, z)yz + c(x, z) \), where \( \ell \) and \( c \) are homogeneous linear and cubic, respectively. The coefficient of \( y^2z \) will be nonzero, and can be normalized to 1. Then \( f = 0 \) is a quadratic equation in \( y \). Completing the square by the substitution \( y \rightarrow y - \frac{1}{2} \ell \) eliminates the linear term, leaving us with \( y^2z + c'(x, z) \). The quadratic term in \( z \) can be eliminated by a substitution \( x \rightarrow x + *z \).

(c) Since \( C \) is smooth, \( a \) and \( b \) aren’t both zero. Unless \( b = 0 \), scaling can be used to replace \( b \) by 1. If \( b = 0 \), one can make \( a = 1 \) instead.