Chapter 5 STRUCTURE OF VARIETIES IN THE ZARISKI TOPOLOGY

5.1 Dimension

The dimension of a variety can be defined either as a transcendence degree or as the maximal length of a chain of closed subvarieties (see 5.1.3). We use transcendence degree to define the dimension, and we show that it gives the same answer as the one that would be obtained with chains of subvarieties.

The dimension of a variety $X$ is the transcendence degree over $\mathbb{C}$ of its function field, and the dimension of a finite-type domain $A$ is the transcendence degree of its fraction field. Thus if $X = \text{Spec} \ A$, then

$$\dim X = \dim A$$

5.1.1. Corollary. If $Y \xrightarrow{\iota} X$ is the inclusion of an open subvariety or if it is an integral morphism, then

$$\dim Y = \dim X.$$

If $C$ is a proper closed subvariety of an affine variety $X$, some regular function on $X$ will vanish on $C$. Because of this, $C$ will have lower dimension than $X$. But it isn’t obvious how much lower its dimension will be. A subtle fact known as Krull’s Theorem helps to determine the drop in dimension.

The codimension of a closed subvariety $C$ of a variety $X$ is the difference $\dim X - \dim C$ of their dimensions.

5.1.2. Krull’s Principal Ideal Theorem. Let $X = \text{Spec} \ A$ be an affine variety of dimension $n$, and let $f$ be a nonzero element of $A$. Every irreducible component of the zero locus $V_X(f)$ has codimension 1.

proof. Step 1: The case of affine space. Let $A$ be the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$, let $X = \text{Spec} \ A$, and let $f$ be a nonzero element of $A$. When we factor $f$ into irreducible polynomials, say $f = f_1 \cdots f_k$, then $\mathbb{C}[x]$ is a unique factorization domain, the ideals $(f_i)$ will be prime ideals. The irreducible components of the zero locus $V_X(f)$ will be the zero sets $V_X(f_i)$. We may assume that $f$ is irreducible.

We adjust coordinates so that $f$ becomes a monic polynomial in $x_n$ with coefficients in $\mathbb{C}[x_1, \ldots, x_{n-1}]$, say $f = x_n^k + c_{k-1}x_n^{k-1} + \cdots + c_0$ (Lemma 4.2.8). Then $A = A/(f)$ will be integral over $\mathbb{C}[x_1, \ldots, x_{n-1}]$, so it will have transcendence degree $n - 1$, and $V_X(f) = \text{Spec} \ A$ will have codimension 1.

We now consider the general case. We suppose that $V_X(f)$ has a component $D$ of dimension $k$, and we show that $k = n - 1$. 

Step 2: Let $Z$ be the union of the components of $V_X(f)$ distinct from $D$. We eliminate $Z$ by localizing. We choose an element $s$ in $A$ that is identically zero on $Z$, but not identically zero on $D$. Then the localization $X_s$ contains points of $D$, but no point of $Z$. The dimensions of the localizations $X_s$ and $D_s$ will be the same as the dimensions of $X$ and $D$. We replace $X$ by $X_s$ and $D$ by $D_s$.

Step 3: We suppose $D = V_X(f)$ is irreducible, and that it has dimension $k$, and we apply the Noether Normalization Theorem. There is a polynomial subring $R = \mathbb{C}[x_1, \ldots, x_n]$ over which $A$ is a finite module. Let $F$ and $K$ be the fraction fields of $R$ and $A$, respectively. Then $K$ is a finite extension of $F$ that we may embed into a Galois extension $K_1$ of $F$. Let $A_1$ be the integral closure of $A$ in $K_1$. Then $A_1$ is also the integral closure of $R$ in $K_1$. Let $S = \text{Spec } R$, $X_1 = \text{Spec } A_1$, and let $W_1$ be the zero loci of $f$ in $X_1$. Since the morphism $X_1 \to X$ is integral, $W_1$ maps surjectively to $D$. Every component $D_i$ of $W_1$ lies over a subvariety of $D$, though not necessarily over $D$ itself. So every component of $W_1$ will have dimension at most $k$, and at least one component will have dimension equal to $k$. We replace $A$, $X$, and $D$ by $A_1$, $X_1$, and $W_1$. The set $W_1$ isn’t necessarily irreducible, but the important point is that all of its components have dimension at most $k$, and that at least one has dimension $k$. So we may assume that $K$ is a Galois extension of $F$. We drop the subscript 1.

Step 4: Let $G$ be the Galois group of $K$ over $F$, and let $f_1, \ldots, f_r$ be the $G$-orbit of $f$, with $f = f_1$. The elements $f_i$ are integral over $R$, and the product $g = f_1 \cdots f_r$ is in $F$. Since $R$ is integrally closed, $g$ is in $R$. We will show that the zero locus $V = V_S(g)$ is the image of $W = V_X(f)$. Then since $X$ is integral over $S$, every component $C$ of $V$ will be the image of a component $D$ of $W$, and the dimensions of $C$ and $D$ will be equal. According to Step 1, every component of $V$ has codimension 1. Therefore $D$ and $C$ have dimension $n - 1$. This will show that $k = n - 1$.

Let $p$ be a point of $V$, and let $q$ be a point of $X$ that lies over $p$. Then $g(q) = g(p) = 0$. Since $g = f_1 \cdots f_r$, $f_i(q) = 0$ for some $i$. The elements $f_i$ form an orbit, so $f_i = \sigma f_1$ for some $\sigma$ in $G$. Let $\pi_q$ denote the homomorphism $A \to R$ that corresponds to $q$, as usual. Then (2.8.2)

$$0 = f_i(q) = \pi_q(f_i) = \pi_q(\sigma f_1) = \pi_q(\sigma f_1) = f_1(q\sigma)$$

Since $q$ lies over $p$, so does $q' = q\sigma$. Since $f(q') = 0$, $q'$ is in $W$. Therefore $p$ is in the image of $W$. \hfill \Box

(5.1.3) chains of subvarieties

A chain of subvarieties of $X$ of length $k$ is a strictly decreasing sequence

$$C_0 > C_1 > C_2 > \cdots > C_k$$

of closed subvarieties. This chain is maximal if it cannot be lengthened by inserting another closed subvariety, which means that $C_0 = X$, that for $i < k$ there is no closed subvariety $\hat{C}$ with $C_i > \hat{C} > C_{i+1}$, and that $C_k$ is a point.

Maximal chains in $\mathbb{P}^2$ have the form $\mathbb{P}^2 > C > p$, where $C$ is a plane curve and $p$ is a point. The chain

$$\mathbb{P}^n > \mathbb{P}^{n-1} > \cdots > \mathbb{P}^0$$

in which $\mathbb{P}^k$ is the set of points $(x_0, \ldots, x_k, 0, \ldots, 0)$ of $\mathbb{P}^n$ is a maximal chain of closed subvarieties of $\mathbb{P}^n$.

(5.1.5) Lemma. Let $X'$ be an open subvariety of a variety $X$. There is a bijective correspondence between chains $C_0 > \cdots > C_k$ of closed subvarieties of $X$ such that $C_k \cap X' \neq \emptyset$ and chains $C_0' > \cdots > C_k'$ of closed subvarieties of $X'$, defined by $C_0' = C_0 \cap X'$. Given a chain $C_i'$ in $X'$, the corresponding chain in $X$ consists of the closures $C_i$ of the varieties $C_i'$ in $X$.

Proof. Suppose given a chain $\{C_i\}$ and that $C_k \cap X' \neq \emptyset$. Then the intersections $C_i = C_i \cap X'$ are nonempty for all $i$, so they are dense open subsets of the irreducible closed sets $C_i$ (2.7.6). The closure of $C_i'$ is $C_i$. Since $C_i$ is irreducible and $C_i > C_{i+1}$, it is also true that $C_i'$ is irreducible and $C_i' > C_{i+1}'$. Therefore $C_0' > \cdots > C_k'$ is a chain of closed subsets of $X'$. Conversely, if $C_0' > \cdots > C_k'$ is a chain in $X'$, the closures in $X$ form a chain in $X$ (see Corollary 2.7.3). \hfill \Box

(5.1.6) Lemma. A closed subvariety $C$ of a variety $X$ has codimension 1 if and only if $X > C$ and there is no closed subvariety $\hat{C}$ such that $X > \hat{C} > C$.
proof. Say that \( \dim X = n \). As Lemma 5.1.5 shows, we may assume \( X \) affine. We may also assume that \( X > C \). Then there will be a regular nonzero function \( f \) that vanishes on \( C \). Since \( C \) is irreducible, it will be contained in a component \( \tilde{C} \) of the zero locus of \( f \), and by Krull’s Theorem, \( \tilde{C} \) will have codimension 1. If \( \tilde{C} \) has codimension greater than 1, then \( X > \tilde{C} > C \). For the converse, suppose that there is a closed subvariety \( \tilde{C} \) of \( X \) such that \( X > \tilde{C} > C \). Then \( \tilde{C} \) will have codimension at least 1. We apply Krull’s Theorem to \( \tilde{C} \). There will be a nonzero regular function \( g \) on \( \tilde{C} \) that vanishes on \( C \), and then \( \tilde{C} \) will be contained in a component of the zero locus of \( g \), which will have codimension 1 in \( \tilde{C} \). Then \( \tilde{C} \) will have codimension at least 2 in \( X \). \( \square \)

5.1.7. Corollary. Every proper closed subvariety of a variety \( X \) is contained in a closed subvariety of codimension 1.

5.1.8. Theorem. Let \( X \) be a variety of dimension \( n \). All chains of closed subvarieties of \( X \) have length at most \( n \), and all maximal chains have length \( n \).

proof. Induction allows us to assume the theorem true for a variety of dimension less than \( n \), and the case \( n = 0 \) is trivial.

Let \( X \) be a variety of dimension \( n \). Lemma 5.1.3 shows that we may assume \( X \) affine, say \( X = \text{Spec} A \). Let \( C_0 > C_1 > \cdots > C_k \) be a chain in \( X \). We are to show that \( k \leq n \) and that \( k = n \) if the chain is maximal. We can insert closed subvarieties into the chain where possible, so we may assume that \( C_0 = X \) and that \( C_1 \) has codimension 1, and dimension \( n - 1 \).

By induction, the length of the chain \( C_1 > \cdots > C_k \), which is \( k - 1 \), is at most \( n - 1 \), and is equal to \( n - 1 \) if it is a maximal chain in \( C_1 \). Lemma 5.1.6 shows that this happens if and only if the given chain \( C_0 > C_1 > \cdots > C_k \) is maximal in \( X \). \( \square \)

5.1.9. Corollary. If \( Y \) is a proper closed subvariety of a variety \( X \), then \( \dim Y < \dim X \).

Theorem 5.1.8 can also be stated in terms of prime ideals. A chain (5.1.10) in \( X = \text{Spec} A \) will correspond to an increasing chain of prime ideals of \( A \) of length \( k \), a prime chain. This prime chain is maximal if it cannot be lengthened by inserting another prime ideal, which means that \( P_0 \) is the zero ideal, that for \( i < k \) there is no prime ideal \( \tilde{P} \) with \( P_i < \tilde{P} < P_{i+1} \), and that \( P_k \) is a maximal ideal. In terms of prime chains, Theorem 5.1.8 is this:

5.1.10. Corollary. Let \( A \) be a finite-type domain of transcendence degree \( n \). All prime chains in \( A \) have length at most \( n \), and all maximal prime chains have length equal to \( n \).

For example, the polynomial algebra \( \mathbb{C}[x_1, \ldots, x_n] \) in \( n \) variables has transcendence degree \( n \), and therefore it has dimension \( n \). The chain of prime ideals

\[
\begin{align*}
P_0 < P_1 < P_2 < \cdots < P_k,
\end{align*}
\]

is a maximal prime chain.

A prime ideal \( P \) of a noetherian domain has codimension 1 if it is not the zero ideal, and if there is no prime ideal \( \tilde{P} \) such that \( (0) < \tilde{P} < P \). Krull’s Theorem shows that the prime ideals of codimension 1 in the polynomial algebra \( \mathbb{C}[x_1, \ldots, x_n] \) are the principal ideals generated by irreducible polynomials.

5.2 Localization II

locring

If \( s \) is a nonzero element of a domain \( A \), the simple localization \( A_s \) is the ring obtained by adjoining an inverse of a nonzero element \( s \). To work with the inverses of finitely many nonzero elements, one may simply adjoin the inverse of their product.

For working with an infinite set of inverses, the concept of a multiplicative system is useful. A multiplicative system \( S \) in a domain \( A \) is a subset that consists of nonzero elements, is closed under multiplication, and contains 1. If \( S \) is a multiplicative system, the ring of \( S \)-fractions \( AS^{-1} \). It is also called a localization of \( A \). This localization is the ring obtained by inverting all elements of \( S \). Its elements are equivalence classes of fractions \( as^{-1} \) with \( a \) in \( A \) and \( s \) in \( S \), the equivalence relation and the laws of composition being the usual ones for fractions.
5.2.1. **Examples.** 
(i) The set consisting of the powers of a nonzero element \( s \) is a multiplicative system. The ring of fractions of this system is the simple localization \( A_s = A[s^{-1}] \).

(ii) When \( S \) is the set of all nonzero elements of \( A \), the localization \( AS^{-1} \) is the field of fractions of \( A \).

(iii) Let \( P \) be a prime ideal of \( A \). The complement of \( P \) in \( A \) is a multiplicative system.

If \( s_1 \) and \( s_2 \) aren’t in \( P \), then because \( P \) is a prime ideal, the product \( s_1 s_2 \) isn’t in \( P \) either. The unit element 1 isn’t in \( P \) because \( P \) isn’t the unit ideal. In fact, an ideal is a prime ideal if and only if its complement is a multiplicative system. □

5.2.2. **Proposition.** Let \( S \) be a multiplicative system in a domain \( A \), and let \( A' \) denote the localization \( AS^{-1} \).

(i) Let \( I \) be an ideal of \( A \). The extended ideal \( IS^{-1} \) is the set \( IS^{-1} \) whose elements are classes of fractions \( xs^{-1} \), with \( x \) in \( I \) and \( s \) in \( S \). The extended ideal is the unit ideal if and only if \( I \) contains an element of \( S \).

(ii) Let \( J \) be an ideal of \( A' \) and let \( I \) denote its contraction \( J \cap A \). The extended ideal \( IS^{-1} \) is equal to \( J \):

\[
\text{(iii) If } P \text{ is a prime ideal of } A \text{ and if } P \cap S \text{ is empty, the extended ideal } P' = PA' \text{ is a prime ideal of } A', \text{ and its contraction } P' \cap A = P. \text{ If } P \cap S \text{ isn’t empty, the extended ideal is the unit ideal.} \quad \square
\]

Thus \( J = \text{extend(contract}(J)) \), and \( I \subset \text{contract(extend}(I)) \).

Part (iii) tells us that prime ideals of \( A' \) correspond bijectively to prime ideals of \( A \) that don’t meet \( S \).

5.2.3. **Corollary.** A localization \( AS^{-1} \) of a noetherian domain \( A \) is noetherian. □

5.2.4. **Note.** An elementary, but important, principle for working with fractions is that any finite sequence of computations in a localization \( AS^{-1} \) will involve only finitely many denominators, and can therefore be done in a simple localization \( A_s \), where \( s \) is a common denominator for the fractions that occur. The next proposition makes use of this principle.

5.2.5. **Proposition.** Let \( A \subset B \) be finite-type domains. There is a nonzero element \( s \) in \( A \) such that \( B_s \) is a finite module over a subring of the form \( A_s[y_1, \ldots, y_r] \), whose elements are polynomials with coefficients in \( A_s \).

**proof.** Let \( S \) be the set of nonzero elements of \( A \), so that \( AS^{-1} \) is the fraction field \( K \) of \( A \), and let \( B_K = BS^{-1} \). Then \( B_K \) is a finite-type \( K \)-algebra. It is generated as \( K \)-algebra by a set \( \beta_1, \ldots, \beta_r \) that generates the finite-type \( \mathbb{C} \)-algebra \( B \). The Noether Normalization Theorem tells us that \( B_K \) is a finite module over a polynomial subring \( K[y_1, \ldots, y_r] \). Then \( B \) is an integral extension of this polynomial ring.

Any element \( b \) of \( B \) will be in \( B_K \), and therefore it will be the root of a monic polynomial of the form

\[
f(x) = x^n + c_{n-1}(y)x^{n-1} + \cdots + c_0(y) = 0
\]

whose coefficients \( c_j(y) \) are elements of \( K[y] \). Each \( c_j(y) \) is a combination of finitely many monomials in \( y \), with coefficients in \( K \). If \( s \in A \) is a common denominator for those coefficients, then \( c_j(x) \) will have coefficients in \( A_s[y] \).

We may choose a common denominator \( s \) for any finite set of elements of \( K \). Since the generators \( \beta_1, \ldots, \beta_r \) of the algebra \( B \) are integral over \( k[y] \), we may choose \( s \) so that all of those elements are integral over \( A_s[y] \). The algebra \( B_s \) is generated over \( A_s \) by those elements, so it will be an integral extension of \( A_s \). □

5.2.6. **local rings**

A local ring \( R \) is a noetherian ring that contains just one maximal ideal \( M \). A local ring will have a quotient field \( k = R/M \), called the residue field of \( R \).

We make a few comments about local rings here, though we will use mainly some special local rings, discrete valuation rings, that will be discussed in the next section.

5.2.7. **Lemma.** A noetherian ring \( R \) is a local ring if and only if the set of elements of \( R \) that aren’t units is an ideal.
proof. If \( R \) is a local ring with maximal ideal \( M \) and \( s \) is an element of \( R \) not in \( M \), then \( s \) isn’t in any maximal ideal, so it is a unit. And because \( M \) isn’t the unit ideal, its elements aren’t units. Conversely, suppose that the set \( M \) of non-units of a ring \( R \) is an ideal. Then the unit ideal is the only larger ideal, so \( M \) is a maximal ideal. Moreover, if an ideal of \( R \) isn’t the unit ideal, then its elements aren’t units, so it is contained in \( M \). So \( M \) is the only maximal ideal.

Let \( P \) be a prime ideal of a noetherian domain \( A \), and let \( S \) be the complement of \( P \). The ring of \( S \)-fractions is a local ring called the \textit{local ring of} \( A \) at \( P \). There are various notations for this local ring, one being \( A_P \), though this notation conflicts badly with the notation \( A_n \) for \( A[z^{-1}] \). The elements of \( P \) are the ones that are \textit{not} inverted in the local ring \( A_P \), while in \( A_n \) it is the element \( s \) that \textit{is} inverted. To make matters even more confusing: If \( p \) is a point of an affine variety \( X = \text{Spec} \ A \), the local ring of \( A \) at the maximal ideal \( m_p \) is also often denoted by \( A_p \). Thus if \( S = A - m_p \), then \( A_{m_p} \) or \( A_p \), and \( AS^{-1} \) are three notations for the same local ring.

\[ \text{5.2.8. Corollary. There is a bijective correspondence between prime ideals of the localization of \( A \) at \( P \) and prime ideals of \( A \) that are contained in \( P \).} \]

\[ \text{5.2.9. Example. (localization of the polynomial ring} \ A = \mathbb{C}[x, y] \text{)} \]

Let \( m \) be the maximal ideal of \( A \) at the origin \( p \) in \( \mathbb{A}^2 = \text{Spec} \ A \). A polynomial \( g \) is in \( m \) if and only if \( g(0, 0) = 0 \). So the elements of the local ring \( A_m \) are fractions of polynomials \( f g^{-1} \), with \( g(0, 0) \neq 0 \).

The prime ideals of \( A_m \) are the extensions of the prime ideals of \( A \) that are contained in \( m \). Those prime ideals are: the zero ideal, the ideal \( m \) itself, and the principal ideals \( fA \) generated by irreducible polynomials such that \( f(0, 0) = 0 \) – the ideals of affine curves \( C \) that contain the origin.

Let’s denote the set of prime ideals of \( A_m \) by \( X_p \). When one passes from \( X \) to \( X_p \) all points except the origin \( p \) and all curves that don’t contain \( p \) disappear. If a curve \( C \) contains \( p \), all points except \( p \) are gone in \( X_p \), but the origin and what is left of the curve remain. Intuitively, one thinks of \( X_p \) as a neighborhood of the origin in the plane.

\[ \text{5.2.10. Local Nakayama Lemma. Let } R \text{ be a local ring with maximal ideal } m \text{ and residue field } k = R/m \text{ and let } V \text{ be a finite } R\text{-module.} \]

(i) Let \( \overline{V} = V/mV \). If \( \overline{V} = 0 \), then \( V = 0 \).

(ii) Let \( S = \{ v_1, ..., v_r \} \) be a set of elements of \( V \), whose residues \( \overline{v}_1, ..., \overline{v}_r \) span \( \overline{V} \). Then \( S \) spans \( V \).

\[ \text{proof. (i) If } \overline{V} = 0, \text{ then } V = mV, \text{ and there is an element } z \in m \text{ such that } 1 - z \text{ annihilates } V. \text{ Then } 1 - z \text{ is not in } m, \text{ so it is a unit. A unit annihilates } V, \text{ and therefore } V = 0. \]

(ii) Let \( W \) be the submodule of \( V \) spanned by \( S \). Let the quotient \( C = V/W \) be a finite \( R \)-module. When we tensor the exact sequence

\[ 0 \to W \to V \to C \to 0 \]

with \( k \), we obtain an exact sequence

\[ \overline{W} \to \overline{V} \to \overline{C} \to 0 \]

(See Proposition \[9.7.6\]) We are given that the image of \( W \) generates \( \overline{V} \). Therefore \( \overline{C} = 0 \), and by (i), \( C = 0 \). Therefore \( W = V \). \]

\[ \text{5.2.11. Corollary. Let } R \text{ be a local ring with maximal ideal } m \text{ and residue field } k. \text{ If the residues of a set } S = \{ v_1, ..., v_r \} \text{ of elements of } m \text{ span the } k\text{-vector space } m/m^2, \text{ then } S \text{ spans } m. \]

\[ \text{5.3 Valuation Rings} \]

A local domain \( R \) with maximal ideal \( M \) has \textit{dimension one} if \( (0) \) and \( M \) are distinct, and are the only prime ideals of \( R \), or if \( (0) < M \) is a maximal prime chain in \( R \). In this section, we describe the normal local domains of dimension one. They are \textit{discrete valuation rings}. 5
Let $K$ be a field, and let $K^\times = K - \{0\}$. A discrete valuation on $K$ is a surjective map
\[
\nu : K^\times \rightarrow \mathbb{Z}
\]
with these properties:

- $\nu(ab) = \nu(a) + \nu(b)$, i.e., $\nu$ is a group homomorphism, and
- $\nu(a + b) \geq \min\{\nu(a), \nu(b)\}$, if $a + b \neq 0$.

The word “discrete” refers to the fact that $\mathbb{Z}^+$ is a discrete ordered group. Other valuations exist and they are interesting, but they seem less important, and we won’t use them. So to shorten terminology, we will refer to a discrete valuation simply as a valuation.

Let $k$ be a positive integer. If $\nu$ is a valuation and if $\nu(\alpha) = k$, then $k$ is the order of zero of $\alpha$, and if $\nu(\alpha) = -k$, then $k$ is the order of pole of $\alpha$ (with respect to the valuation).

5.3.2. Lemma. Let $\nu$ be a valuation on a field $K$ that contains the complex numbers. Then $\nu(c) = 0$ for all nonzero complex numbers $c$.

proof. This is true because $\mathbb{C}$ contains $n$th roots. The first property of a valuation shows that if $\gamma^r = c$, then $\nu(\gamma) = \nu(c)/n$. The only integer that is divisible by every integer $r$ is zero.

The valuation ring $R$ associated to a valuation $\nu$ on a field $K$ consists of the elements of $K$ whose values are non-negative, together with zero:

\[
R = \{a \in K^\times \mid \nu(a) \geq 0\} \cup \{0\}.
\]

Valuation rings are often called “discrete valuation rings”, but since we have dropped the word discrete from the valuation, we drop it from the ring too.

5.3.4. Proposition. Let $R$ be the valuation ring of a valuation $\nu$ on a field $K$.

(i) $R$ is a local domain. Its maximal ideal $M$ is the set of elements with positive value:

\[
M = \{a \in K \mid \nu(a) > 0\}.
\]

This is a principal ideal. It is generated by any element $x$ such that $\nu(x) = 1$.

(ii) The units of $R$ are the elements with value zero. Every nonzero element of $K$ has the form $x^k u$, where $u$ is a unit and $k$ is an integer.

(iii) Let $N$ be an $R$-submodule of $K$, and assume that $0 < N < K$. Then $N = x^k R$ for some $k$ in $\mathbb{Z}$. The nonzero ideals of $R$ are the powers $M^k$ of $M$, with $k \geq 0$. Therefore $R$ is noetherian.

(iv) If $R$ is a proper subring of a ring $R'$, then $R' = K$. There is no ring $R'$ such that $R < R' < K$.

proof. (ii) Let $z$ be a nonzero element of $K$ and let $\nu(z) = k$. Then, with $x$ as in (i), $x^{-k} z$ is a unit in $R$, so $zR = x^k R$.

(iii) Let $N$ be a nonzero submodule of $K$ and suppose that the values of the elements of $N$ are bounded below. Then if $k$ is the greatest lower bound of those values, $N = x^k R$. If the values of the elements are not bounded below, then $N$ contains $x^k R$ for every $k$, and $N = K$.

(iv) This follows from (iii).

5.3.5. Example. The valuations of the field of rational functions in one variable correspond bijectively to points of the projective line $\mathbb{P}^1$.

proof. Let $K$ denote the field $\mathbb{C}(t)$ of rational functions, and let $a$ be a complex number. To define the valuation $\nu_a$ that corresponds to the point $t = a$ of $\mathbb{P}^1$, we write a nonzero polynomial as $p = (t - a)^k h$, where $t - a$ doesn’t divide $h$, and we define, $\nu_a(p) = k$. We define the value of a nonzero rational function $p/q$ to be $\nu_a(p/q) = \nu_a(p) - \nu_a(q)$. You will be able to check that with this definition, $\nu_a$ becomes a valuation. The valuation that corresponds to the point at infinity of $\mathbb{P}^1$ is obtained by working with $t^{-1}$ in place of $t$. 
The valuation ring associated to the valuation $v_a$ is the localization of $\mathbb{C}[t]$ at the point $t = a$. Its elements are fractions $p/q$ such that $t - a$ doesn’t divide $q$.

To complete the proof, we show that every valuation $v$ of the field $K = \mathbb{C}(t)$ corresponds to a point of $\mathbb{P}^1$. Let $R$ be the valuation ring of a valuation $v$. If $v(t) < 0$, then $v(t^{-1}) > 0$. In that case we replace $t$ by $t^{-1}$. So we may assume that $t$ is an element of $R$, and therefore that $\mathbb{C}[t] \subset R$.

The maximal ideal $M$ of $R$ isn’t zero. It contains a nonzero element of $K$, a fraction of polynomials. Since $\mathbb{C}[t] \subset R$, we can clear the denominator in this fraction, while staying in $M$. So $M$ contains a nonzero polynomial $f$. Since $M$ is a prime ideal, it contains an irreducible factor of $f$, of the form $t - a$ for some complex number $a$. Then $t - a$ is in $M$. But if $c \neq a$, then $c - a$ isn’t in $M$, and so $t - c$ isn’t in $M$ either. It is a unit of $R$. It follows that $R$ contains the localization $R_0$ of $\mathbb{C}[t]$ at the point $t = a$, which is a valuation ring. There is no ring properly containing $R_0$ except $R$, so $R_0 = R$.

5.3.6. Theorem. (i) A local domain $R$ whose maximal ideal $M$ is a nonzero principal ideal is a valuation ring.

(ii) The discrete valuation rings are the normal local domains of dimension 1.

proof. (i) Say that $M$ is a nonzero principal ideal, say $xR$. Let $y$ be a nonzero element of $R$ and let $x^k$ be the largest power of $x$ that divides $y$. Then $y = ux^k$, where $u$ is in $R$ but not in $M = xR$. Since $R$ is a local ring, $u$ is a unit. Then any nonzero element $z$ of the fraction field $K$ of $R$ has the form $z = vx^r$ where $r$ is a positive or negative integer and $v$ is a unit. This is seen by writing the numerator and denominator of a fraction in such a form and dividing. The valuation whose valuation ring is $R$ is defined by $v(z) = r$, where $r$ is as above. If $z = v_i x^{r_i}$, $i = 1, 2$, where $v_i$ is a unit and $r_1 \leq r_2$, then $z_1 + z_2 = \alpha x^{r_1}$, where $\alpha = v_1 + v_2 x^{r_2 - r_1}$ is an element of $R$. Therefore $v(z_1 + z_2) \geq r_1 = \min\{v(z_1), v(z_2)\}$. The requirements for a valuation are satisfied.

(ii) The normalization $R'$ of a discrete valuation ring $R$ is a finite $R$-module contained in the fraction field $K$. Since $K$ isn’t a finite $R$-module, Proposition 5.3.3(iii) shows that $R = R'$.

Conversely, let $R$ be a normal local domain of dimension 1. We show that $R$ is a valuation ring by showing that the maximal ideal of $R$ is a principal ideal. Let $\alpha$ be a nonzero element of $M$. Because $R$ has dimension 1, $M$ is the only prime ideal that contains $\alpha$, so $M$ is the radical of the principal ideal $\alpha R$, and $M' \subset \alpha R$ for large $r$. Let $r$ be the smallest such integer. Then $r > 0$. If $r = 1$, then $M = \alpha R$, so $M$ is a principal ideal. If $r > 1$, there is an element $\beta$ in $M'^{-1}$ such that $\beta \notin \alpha R$, but $\beta M \subset \alpha R$. Let $\gamma = \beta/\alpha$. Then $\gamma \notin R$, but $\gamma M \subset R$. Since $M$ is an ideal, multiplication by an element of $R$ carries $\gamma M$ to itself. So $\gamma M$ is an ideal too. Since $R$ is a local ring with maximal ideal $M$, either $\gamma M \subset M$ or $\gamma M = R$. If $\gamma M \subset M$, the lemma elow shows that $\gamma$ is integral over $R$. This is impossible because $R$ is normal and $\gamma \notin R$. Therefore $\gamma M = R$. Then $M = \gamma^{-1} R$. This implies that $\gamma^{-1}$ is in $R$, and that $M$ is a principal ideal.

5.3.7. Lemma. Let $I$ be a nonzero ideal of a noetherian domain $A$, and let $B$ be a domain that contains $A$. An element $\gamma$ of $B$ such that $\gamma I \subset I$ is integral over $A$.

proof. This is the Nakayama Lemma again. Because $A$ is noetherian, $I$ is finitely generated. Let $v = (v_1, \ldots, v_d)^t$ be a vector whose entries generate $I$. The hypothesis $\gamma I \subset I$ allows us to write $\gamma v = \sum p_j v_j$ with $p_j$ in $A$, or in matrix notation, $\gamma v = P v$. Let $p(t)$ be the characteristic polynomial of $P$. Then $p(\gamma) v = 0$. Since $I \neq 0$, at least one $v_i$ is nonzero. Therefore, since $A$ is a domain, $p(\gamma) = 0$. The characteristic polynomial is a monic polynomial with coefficients in $A$, so $\gamma$ is integral over $A$.

5.4 Smooth Affine Curves

A curve is a variety of dimension 1. Its proper closed subsets are the finite sets.

Let $X = \text{Spec } A$ be an affine curve. A rational function is regular on $X$ if and only if it is regular at every point $p$, which means that it is in every every local ring $A_p$. But we also know that $\alpha$ is regular if and only if it is an element of $A$ (Proposition 5.3.3). Therefore the coordinate ring $A$ of an affine curve $X = \text{Spec } A$ is the intersection of its localizations:

$$ A = \bigcap A_p \quad \text{(in } K) $$

In fact, this is true for any affine variety. consequence is that a domain $A$ is normal if and only if all of its localizations $A_p$ are normal. (This follows from Lemma 5.3.3(ii)).
A point \( p \) of a curve \( X \) is a smooth point if the local ring at \( p \) is a valuation ring, and a curve is smooth if all of its points are smooth. Thus an affine curve \( X \) is smooth if and only if its coordinate algebra is a normal domain (Theorem 5.3.6).

If a curve \( X \) is smooth at \( p \), we denote the corresponding valuation by \( \nu_p \). The zeros \( Z \) and the poles \( P \) of a rational function \( \alpha \) on a smooth curve \( X \) are defined as the points \( p \) at which \( \alpha \) has a zero or a pole, with respect to the valuation \( \nu_p \).

### 5.4.2. Proposition

Let \( X = \text{Spec } A \) be a smooth affine curve. The localizations \( A_p \) of \( A \) at the points \( p \) of \( X \) are the valuation rings of the fraction field \( K \) that contain \( A \).

**proof.** First, the localization \( A_p \) at a point \( p \) is a valuation ring that contains \( A \) (Theorem 5.3.6). Let \( R \) be a valuation ring of \( K \) that contains \( A \), let \( v \) be the associated valuation, and let \( M \) be the maximal ideal of \( R \). The intersection \( M \cap A \) is a prime ideal of \( A \). Since \( A \) has dimension 1, the zero ideal is the only prime ideal of \( A \) other than the maximal ideals. To verify that \( M \cap A \) isn’t the zero ideal, we choose a nonzero element \( \alpha \in M \), and write it as a fraction \( a/b \), with \( a \) and \( b \) in \( A \). Then \( v(a) \geq v(\alpha) > 0 \), so \( a = bo \) is a nonzero element of \( M \cap A \).

Since \( M \cap A \) isn’t zero, it is the maximal ideal \( m_p \) of \( A \) corresponding to a point \( p \) of \( X \). The elements of \( A \) not in \( m_p \) aren’t in \( M \) either, and they are invertible in \( R \). Therefore the local ring \( A_p \), at \( p \), which is a valuation ring, is contained in \( R \). So \( A = R \) (5.3.4 (iii)).

### 5.4.3. Corollary

Let \( X \) be a smooth curve, not necessarily affine, with function field \( K \). Morphisms \( X \to \mathbb{P}^n \) correspond bijectively to points of \( \mathbb{P}^n \) with values in \( K \).

**proof.** Let \( (\alpha_0, ..., \alpha_n) \) be a point of \( \mathbb{P}^n \) with values in \( K \). Proposition ?? tells us that \( \alpha \) determines a morphism \( X \to \mathbb{P}^n \) if and only if, for every point \( p \) of \( X \), there is an index \( i \) such that the functions \( \alpha_j/\alpha_i \) are regular at \( p \) for every \( j \). The functions \( \alpha_j/\alpha_i \) will be regular at \( p \) when \( i \) is chosen so that the order of zero \( \nu_p(\alpha_i) \) of \( \alpha_i \) at \( p \) is minimal.

This Corollary isn’t true in dimension greater than one. If \( X \) is the affine plane \( \text{Spec } \mathbb{C}[x, y] \), its function field \( K \) is the field \( \mathbb{C}(x, y) \) of rational functions. The pair of functions \( x, y \) defines a point of \( \mathbb{P}^1 \) with values in \( K \), but not a morphism \( X \to \mathbb{P}^1 \). There is no way to extend the map to the origin.

### 5.4.4. Lemma

Let \( X \) be a smooth affine curve with coordinate algebra \( A \) and function field \( K \), and let \( p \) be a point of \( X \). There exists an element \( \alpha \in K \) with pole of order 1 at \( p \) and no other pole.

If the maximal ideal \( m_p \) of \( A \) at \( p \) is a principal ideal, a generator \( t \) will have \( p \) as its only zero. Then \( t^{-1} \) will have \( p \) as its only pole, and it will have no zeros. If \( m_p \) isn’t a principal ideal, the element we are looking for will have some zeros as well as its single pole.

**proof of the lemma.** Let \( R \) denote the local ring \( A_p \) at \( p \), and let \( t \) be an element of \( A \) that generates the maximal ideal of \( R \). Then \( t \) will have a zero of order 1 at \( p \), and because \( X \) has dimension one, it will have finitely many other zeros, say \( q_1, ..., q_r \). There is an element \( z \) of \( A \) that is zero at \( q_1, ..., q_r \) but not zero at \( p \). Then for large \( n \), \( z^n t^{-1} \) will be an element of \( K \) with a pole of order 1 at \( p \), and no other pole.

### 5.4.5. Proposition

Let \( X = \text{Spec } A \) be a smooth affine curve, and let \( m \) be the maximal ideal of \( A \) at a point \( p \) of \( X \). If \( I \) is an ideal whose radical is \( m \), then \( I \) is a power \( m^k \) of \( m \).

**proof.** Let \( v \) be the valuation corresponding to the point \( p \), and let \( R \) be the associated valuation ring, the local ring of \( A \) at \( p \). The nonzero ideals of \( R \) are powers of its maximal ideal \( M \).

The maximal ideal \( m \) consists of the elements \( a \) of \( A \) with \( v(a) \geq 1 \). Therefore \( m^r \) contains elements that have value \( r \), and all nonzero elements of \( m^r \) have value at least \( r \). Let \( k \) be the minimal value \( v(x) \) among the nonzero elements \( x \) of \( I \). Every nonzero element of \( I \) has value at least \( k \). We will show that \( I \) is the set of all elements \( y \) of \( A \) with \( v(y) \geq k \). Since we can apply the same reasoning to \( m^k \), it will follow that \( I = m^k \).

Let \( y \) be a nonzero element of \( A \) with \( v(y) \geq k \). Then since \( v(uy) \geq v(x) \), \( x \) divides \( y \) in \( R \). So we may write \( y \) in the form \( y = s^{-1} ax \), where \( s, a \) are in \( A \), and \( s \notin m \). The element \( s \) will vanish at a finite set of points \( q_1, ..., q_r \) distinct from \( p \).

We choose an element \( s' \) of \( A \) that vanishes at \( p \) but not at any of the points \( q_1, ..., q_r \), and we look at the localization \( A_{s'} \). The extended ideal \( mA_{s'} \) is the unit ideal. Since the radical of \( I \) is \( m \), the localized ideal \( I_{s'} \)
is the unit ideal too. Therefore $y$ is in $I_t$. We may write $y = s'^{-n}b$ for some $b \in I$. Since we can replace $s'$ by a power, we may assume that $y = s'^{-1}b$. We now have the two equations

$$sy = ax \quad \text{and} \quad s'y = b$$

among elements of $A$. By our choice, $s'$ and $s$ have no common zeros in $X = \text{Spec} A$. They generate the unit ideal of $A$. Writing $us + u's' = 1$ with $u, u'$ in $A$, we have $y = (us + u's')y = uax + u/b$. The right side of this equation is in $I$, so $y$ is in $I$.

5.4.6. Corollary. Every nonempty open subvariety $X'$ of a smooth affine curve $X$ is a smooth affine curve.

\textit{proof.} A nonempty open subset of a curve is the complement of a finite set, so it will be enough to consider the case of the open set $X'$ obtained by deleting a single point $p$ of $X$. Lemma 5.4.4 tells us that there is an element $\alpha$ in $K$ with a pole at $p$ and no other pole. Let $A_1$ denote the finite type domain $A[\alpha]$. We show that $X_1 = \text{Spec} A_1$ is isomorphic to $X'$.

The inclusion $A \subset A_1$ gives us a morphism $X_1 \to X$. If $q$ is a point of $X$ different from $p$, then $\alpha$ is an element of the local ring $A_q$. Therefore $A_1 \subset A_q$, and so there is a point $q_1$ of $X_1$ that maps to $q$. Since $A_q$ is a valuation ring, $A_q = A_{q_1}$ [5.3.3(iii)]. So $q_1$ is the only point of $X_1$ that lies over $q$. One the other hand, since $\alpha \notin A_p$ but $\alpha \in A_1$, there is no point of $X_1$ lying over $p$. So the map $u$ sends $X_1$ bijectively to $X' = X - \{p\}$. The map is a homeomorphism simply because the proper closed sets in $X_1$ and in $X'$ are the finite sets. To show that the inverse map $X' \to X_1$ is an isomorphism, we must show that if a rational function $\beta$ is regular at a point $q_1$ of $X_1$, then it is regular at $q = v(q_1) = u^{-1}(q_1)$. This is true because the local rings are equal. $\square$

5.4.7. the Jacobian criterion

5.4.8. Proposition. Let $X = \text{Spec} A$ be an affine curve with coordinate algebra $A = \mathbb{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_k)$. A point $p$ of $X$ is smooth if and only if the Jacobian matrix $J = \frac{\partial f_i}{\partial x_j}$ has rank $n - 1$ at $p$.

We leave the proof as an exercise. $\square$

This Jacobian criterion generalizes to higher dimension. An affine variety $X$ of dimension $d$ whose coordinate algebra is presented as $A = \mathbb{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_k)$ is smooth at a point $p$ if and only if the Jacobian matrix $J = \frac{\partial f_i}{\partial x_j}$, evaluated at $p$, has rank $n - d$. However, to apply this criterion, one needs to know the dimension of $X$, and the dimension may not be easy to determine.

5.4.9. Example. The twisted cubic $X$ in $\mathbb{P}^3$ is the curve whose points are $(1, t, t^2, t^3)$ for $t \in \mathbb{C}$ together with the point $(0, 0, 0, 1)$. It is defined by the three homogeneous equations

\begin{align*}
x_0x_3 &= x_1x_2, \\
x_1^2 &= x_0x_2, \\
x_2^2 &= x_1x_3
\end{align*}

The zero locus of the first two equations is the union of the twisted cubic and the line $L : x_0 = x_1 = 0$, and the last equation eliminates all points of the line except $(0, 0, 0, 1)$. The rank of the Jacobian matrix is 2 at all points of $X$, so $X$ is a smooth projective curve. $\square$

5.5 Nodes and Cusps II

We describe nodes and cusps of curves here. Nodes and cusps of plane curves were defined in Chapter ??.

Let $p$ be a singular point of a curve $X$. For simplicity, let’s assume that $X$ is affine and that $p$ is its only singular point. We can achieve this by localizing. Let $k = k(p)$ be the residue field at $p$, and let $\bar{X} = \text{Spec} \bar{A}$ be the normalization of $X$.

5.5.1. Definition. The point $p$ is a node or a cusp if and only if the $A$-module $\epsilon = \bar{A}/A$ has dimension one as complex vector space, i.e., if and only if $\epsilon$ is isomorphic, as module, to the residue field $k = k(p)$. If $\epsilon$ has dimension one, and if there are two points of $\bar{X}$ lying over $p$, then $p$ is called a node, while if there is just one point lying over $p$, then $p$ is called a cusp.
5.5.2. **Lemma.** If \( \epsilon \) has dimension one, the quotient algebra \( \tilde{A}/m\tilde{A} \) has dimension two. Therefore there are at most two points of \( X \) that lie over \( p \).

**proof.** It is convenient to form a diagram in which \( m \) denotes the maximal ideal of \( A \) at \( p \) and \( k = A/m \) denotes the residue field at \( p \):

\[
\begin{array}{cccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & \longrightarrow m & \longrightarrow m\tilde{A} & \longrightarrow m\epsilon & \longrightarrow 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
A & \longrightarrow \tilde{A} & \longrightarrow \epsilon & \longrightarrow 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
k & \longrightarrow \tilde{A}/m\tilde{A} & \longrightarrow \epsilon/m\epsilon & \longrightarrow 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0
\end{array}
\]

The middle row and all three columns are exact. Since \( \epsilon \) is isomorphic to \( k \), \( m\epsilon = 0 \) and therefore \( \epsilon \cong \epsilon/m\epsilon \).

The Snake Lemma, applied to the first two columns, shows that \( m \approx m\tilde{A} \), and that all rows are exact. Then the bottom row shows that \( \tilde{A} \otimes_A k \) has dimension 2.

5.5.4. **Proposition.** (i) Suppose that \( p \) is a node, and let \( q_1 \) and \( q_2 \) be the points of \( \tilde{X} \) over \( p \). Then \( A \) is the subalgebra of \( \tilde{A} \) of elements \( \alpha \) such that \( \alpha(q_1) = \alpha(q_2) \).

(ii) Suppose that \( p \) is a cusp. Let \( \tilde{m} \) be the maximal ideal of \( \tilde{A} \) at the point \( q \) of \( \tilde{X} \) over \( p \). Then \( A \) is the subalgebra \( k + \tilde{m}^2 \) of \( \tilde{A} \).

**proof.** (i) Let \( A' \) be the subalgebra of \( \tilde{A} \) of elements \( \alpha \) such that \( \alpha(q_1) = \alpha(q_2) \). It is obvious that \( A \subset A' \), and that \( A' \subset A \). Since \( \epsilon \) has dimension one, \( A = A' \).

(ii) The only maximal ideal of \( \tilde{A} \) that contains \( m \) is the maximal ideal \( \tilde{m} \) at the single point \( \tilde{p} \) that lies over \( p \). Therefore the radical of the ideal \( m\tilde{A} \) is \( \tilde{m} \), and \( m\tilde{A} \) is a power of \( \tilde{m} \) (Proposition 5.4.5). Since \( \tilde{A}/m\tilde{A} \) has dimension 2, \( m\tilde{A} = \tilde{m}^2 \).

6. Constructible Sets

In this section, \( X \) denotes a noetherian topological space. Every strictly decreasing chain of closed subsets of \( X \) is finite, and every closed subset is a union of finitely many irreducible closed sets.

The intersection \( L = C \cap U \) of a closed subset \( C \) and an open subset \( U \) of \( X \) is a locally closed set. For instance, closed subsets and open subsets are locally closed. A subset is constructible if it is the union of finitely many locally closed sets.

In this section we use the following notation: \( L \) is locally closed, \( C \) is closed, and \( U \) is open.

5.6.1. **Example.** A subset \( S \) of a curve \( X \) is constructible if and only if it is either a finite set or the complement of a finite set. Thus \( S \) is constructible if and only if it is either closed or open, in which case it is locally closed.

The proofs of the next two theorems are elementary topology, but they are confusing enough to require care.

5.6.2. **Theorem.** The set \( \mathcal{S} \) of constructible subsets of a noetherian topological space \( X \) is the smallest set of subsets that contains the open sets and is closed under the three operations of finite union, finite intersection, and complementation.
proof. Let $S_1$ denote the set of subsets that is obtained from the open sets by the three operations mentioned in the statement. Open sets are constructible, and with those operations, one can produce any constructible set from the open sets. So $S \subseteq S_1$.

To show that $S = S_1$, we show that the constructible sets are closed under the three operations. It is obvious that a finite union of constructible sets is constructible. The intersection of two locally closed sets $L_1 = C_1 \cap U_1$ and $L_2 = C_2 \cap U_2$ is locally closed because $L_1 \cap L_2 = (C_1 \cap C_2) \cap (U_1 \cap U_2)$. If $S = L_1 \cup \cdots \cup L_k$ and $S' = L'_1 \cup \cdots \cup L'_{r}$ are constructible sets, the intersection $S \cap S'$ is equal to the union $\bigcup (L_i \cap L'_j)$, so it is constructible.

To show that the complement $S^C$ of a constructible set $S$ is constructible, it suffices to show that the complement of a locally closed set is constructible. For, if $S = L_1 \cup \cdots \cup L_k$, then $S^C = L'_1 \cap \cdots \cap L'_r$, and we know now that intersections of constructible sets are constructible. Let $L$ be the locally closed set $C \cap U$, and let $V = C^C$ and $Y = U^C$ be the complements of $C$ and $U$, respectively. Then $V$ is open and $Y$ is closed. The complement $L^C$ of $L$ is the union $V \cup Y$ of constructible sets, so it is constructible.

\[5.6.3. \text{Theorem.} \ (i) \text{Every constructible set } S \text{ is a union } L_1 \cup \cdots \cup L_k \text{ of locally closed sets } L_i = C_i \cap U_i, \text{ in which the closed sets } C_i \text{ are irreducible and distinct.}\]

\[5.6.4. \text{Proposition.} \ (i) \text{Let } X' \text{ be an open or a closed subvariety of a variety } X. \text{ A subset } S \text{ of } X' \text{ is a constructible subset of } X' \text{ if and only if it is a constructible subset of } X.\]

\[5.6.5. \text{Theorem.} \ Y \xrightarrow{f} X \text{ be a morphism of varieties.}\]

(i) The inverse image of a constructible subset of $X$ is a constructible subset of $Y$.

(ii) The image of a constructible subset of $Y$ is a constructible subset of $X$.

proof. Part (i) follows directly from the fact that $f$ is a continuous map. The proof of (i) is brutal. One hammers away until there is nothing left to do.

Let $S$ be a constructible subset of $Y$. 

Step 1: Suppose that $Y$ is the union of finitely many subvarieties, which may be open or closed, and let $S_j = S \cap Y_j$. Then $S_j$ are constructible and their union is $S$. It suffices to show that the image of each $S_j$ is constructible. Similarly, suppose that $X$ is the union of finitely many open or closed subvarieties $X_i$. Let $Y_i = f^{-1}X_i$ and let $S_i = S \cap Y_i$. It suffices to show that the image of each $S_i$ is constructible. Moreover, Proposition 5.6.4 tells us that the image of $S_i$ is constructible in $X_i$ if and only if it is constructible in $X$.

Step 2: Noetherian induction on $Y$ and on $X$ allows us to assume that the image $f(S)$ is constructible if $S$ is contained in a proper closed subset of $Y$, or if $f(S)$ is contained in a proper closed subset of $X$. Therefore we may assume that $Y$ is the closure of $S$ and that $X$ is the closure of $f(S)$.

When we decompose $X$ into a proper closed subvariety $X_1$ and an open subvariety $X_2$, and we decompose $Y$ by the inverse images $Y_i = f^{-1}X_i$, noetherian induction applies to the map $Y_1 \to X$. Similarly, when we decompose $Y$ in to a proper closed subvariety $Y_1$ and an open subvariety $Y_2$, noetherian induction applies to the map $Y_1 \to X$. In either case, we may ignore $Y_1$. This means that we may replace $X$ by any nonempty open subvariety $X'$ and $Y$ by any nonempty open subvariety of $f^{-1}X'$. We can do this finitely many times.

Step 3: Since $Y$ is the closure of $S$, Theorem 5.2.3(ii) tells us that $S$ contains a nonempty open subset of $Y$. We may replace $Y$ by that subset. So it suffices to show that the image of $Y$ itself is constructible. As Step 2 shows, we may assume that the closure of $f(Y)$ is $X$.

W may still replace $X$ and $Y$ by nonempty open sets, so we may assume that $X$ and $Y$ are affine, say $X = \text{Spec } A$, $Y = \text{Spec } B$, and that the morphism $f$ corresponds to the algebra homomorphism $A \xrightarrow{\varphi} B$. If the kernel of $\varphi$ was a nonzero (prime) ideal $P$, the image of $Y$ would be contained in a proper closed subset of $X$. We have taken care of that case. So $\varphi$ is injective.

Corollary 5.2.5 tells us that, for suitable nonzero $s$ in $A$, $B_s$ is a finite module over a polynomial subring $A_s[y]$. Then both of the maps $Y_s \to \text{Spec } A_s[y]$ and $\text{Spec } A_s[y] \to X_s$ are surjective, so $Y_s$ maps surjectively to $X_s$. When we replace $X$ and $Y$ by $X_s$ and $Y_s$, the map becomes surjective, and we are done. 

5.7 Closed Sets

Limits of sequences are often used to analyze subsets of a topological space. A metric space $Y$ is closed in the classical topology if, whenever a sequence of points in $Y$ has a limit in $\mathbb{R}^n$, the limit is in $Y$. In algebraic geometry one uses morphisms from algebraic curves to $Y$ as a substitute. We use the following notation to state the analogue:

$C$ with point

(5.7.1) $C$ is a smooth affine curve, and $C' = C - q$ is the complement of a point $q$ of $C$.

The (Zariski) closure of $C'$ will be $C$, and we think of $q$ as a limit point. Theorem 5.7.3, which is below, asserts that a constructible subset $Y$ of a variety $X$ is closed if it contains all such limit points. It is based on the next theorem, which states that there are enough curves to do the job.

$W$ enough curves

5.7.2. Theorem. (there are enough curves) Let $Y$ be a constructible subset of a variety $X$, and let $p$ be a point of its closure $\overline{Y}$. There exist a morphism $C \xrightarrow{f} X$ from a smooth curve to $X$ and a point $q$ of $C$ such that $f(q) = p$ and $f(C') \subseteq Y$.

Proof. The method is to use Krull’s Theorem to slice $Y$ down to dimension 1.

If $X = p$, we may take for $f$ the constant morphism from any curve $C$ to $p$. So we may assume that $X$ has positive dimension $d$. Next, we may replace $X$ by any affine open subset that contains $p$, and $Y$ and $\overline{Y}$ by their intersections with that open subset. So we may assume $X$ affine, say $X = \text{Spec } A$.

Since $Y$ is constructible, it is union $L_1 \cup \cdots \cup L_k$ of locally closed sets $L_i = C_i \cap U_i$, where the closed sets $C_i$ are irreducible. The closure of $Y$ is $\overline{Y} = C_1 \cup \cdots \cup C_k$. Since $p$ is in $\overline{Y}$, it is in one of the closed sets $C_i$. We may replace $Y$ by $L_i$ and $X$ by $C_i$, so we may assume that $Y$ is a nonempty open subset of $X$.

Suppose that the dimension $d$ of $X$ is at least two. Let $W = X - Y$ be the complement of $Y$ in $X$. The components of $W$ have dimension at most $d - 1$. We choose a suitable element $\alpha \in A$ such that $\alpha(p) = 0$: We require that $\alpha$ isn’t identically zero on any component of $W$, except for $p$, if $p$ happens to be a component. Krull’s Theorem tells us that every component of the zero locus $V_X(\alpha)$ of $\alpha$ has dimension $d - 1$, and at least one of those components contains $p$. Let $\overline{V}$ be such a component. Since $\alpha$ isn’t identically zero on any
5.8.2. Example. Let $X$, $Y$ and $Z$ be affine lines, let $X \xrightarrow{f} Z$ be the map $z = x^2$, and let $g$ be the map $z = y^2$. The fibred product $X \times_Z Y$ is the closed subset of the affine $x,y$-plane consisting of the diagonal $x = y$ and the antidiagonal $x = -y$.

Rather than discussing schemes, we show that the fibred product of varieties is a (Zariski) closed subset of the product $X \times Y$. This will be enough for our purposes.

5.8.3. Proposition. Let $X \xrightarrow{f} Z$ and $Y \xrightarrow{g} Z$ be morphisms of varieties. The fibred product $X \times_Z Y$ is a closed subset of the product variety $X \times Y$.

proof. Step 1. The graph $\Gamma_f$ of a morphism $X \xrightarrow{f} Z$ is a closed subvariety of $X \times Z$ that is isomorphic to $X$. The graph can be represented as a fibred product by the diagram
\[
\begin{array}{c}
\Gamma_f \longrightarrow X \times Z \\
\downarrow \quad \downarrow f \times id \\
Z_{\Delta} \longrightarrow \Delta \longrightarrow Z \times Z
\end{array}
\]

where \(Z_{\Delta}\) is the diagonal, a closed subset of \(Z \times Z\). The map \(F \times id\) is a morphism, and \(\Gamma_f\) is the inverse image in \(X \times Z\) of the closed subvariety \(Z_{\Delta}\) of \(X \times X\), so it is a closed subset of \(X \times Z\).

The projection of \(\Gamma_f\) to \(X\) is bijective. It is continuous because the projection \(X \times Z \overset{u}{\longrightarrow} X\) is a morphism. Its inverse is obtained using the mapping property of product varieties (Proposition 3.1.16), which gives us a morphism \((id, f) : X \longrightarrow X \times Z\), whose image is \(\Gamma_f\). Therefore \(X\) and \(\Gamma_f\) are homeomorphic. This shows that \(\Gamma_f\) is an irreducible closed set, and therefore a closed subvariety, of \(X \times Z\). The maps \(X \rightarrow \Gamma_f\) and \(\Gamma_f \rightarrow X\) we have described are inverse morphisms, so \(\Gamma_f\) is isomorphic to \(X\).

**Step 2.** Let \(u\) and \(v\) be two morphisms from a variety \(X\) to another variety \(z: X \longrightarrow Z\). The set \(W\) consisting of points \(x\) in \(X\) such that \(u(x) = v(x)\) is closed in \(X\): Let \(W' = \Gamma_u \cap \Gamma_v\) in \(X \times Z\). This is the intersection of \(\Gamma_u\) with the closed set \(\Gamma_v\), so \(W'\) is closed in \(\Gamma_u\). The isomorphism \(\Gamma_u \rightarrow X\) carries \(W'\) to \(W\), so \(W\) is closed in \(X\).

**Step 3:** Completion of the proof. With reference to diagram 5.8.1, \(X \times_Z Y\) is the subset of \(X \times Y\) of points at which the maps \(f_\pi\) and \(g_\pi\) to \(Z\) are equal.

For reference in the next section, we derive a corollary of Theorem 5.7.2.

**5.8.4. Corollary.** (lifting of curves) Let \(W \overset{u}{\longrightarrow} Z\) and \(C \overset{f}{\longrightarrow} Z\) be morphisms of varieties, where \(C\) is a smooth affine curve. If the image \(f(C)\) is contained in the image \(u(W)\), there is a smooth affine curve \(D\) that fits into a diagram of morphisms

\[
\begin{array}{c}
D \overset{f'}{\longrightarrow} W \\
\downarrow \quad \downarrow u \\
C \overset{f}{\longrightarrow} Z
\end{array}
\]

such that \(g\) isn’t a constant map – it isn’t the map from \(D\) to a single point.

In this corollary, we can’t require that the map \(g\) be surjective. Its image will be a nonempty open subset of \(C\).

**proof.** We form the fibred product \(C \times_Z W\). Since \(f(C)\) is contained in \(u(W)\), the projection from \(C \times_Z W\) to \(C\) is surjective. At least one component of \(C \times_Z W\) will map to an open subset of \(C\). We choose such a component, call it \(W'\), and we project \(W'\) to its image, a nonempty open subset \(C'\) of \(C\). Since the map \(g\) we are looking for isn’t required to be surjective, replacing \(C\) by \(C'\) is permissible, and we do that. Then we are looking for a smooth curve \(D\) and morphisms to complete the diagram below:

\[
\begin{array}{c}
D \longrightarrow W' \longrightarrow W \\
\downarrow \quad \downarrow u' \quad \downarrow u \\
C \longrightarrow C \longrightarrow Z
\end{array}
\]

We replace the map \(W \overset{u}{\longrightarrow} Z\) by \(W' \overset{u'}{\longrightarrow} C\). This reduces us to the case that \(f\) is the identity map \(C \rightarrow C\). When we drop the primes from \(W'\) and \(u'\), the problem becomes this: Let \(W \overset{u}{\longrightarrow} C\) be a surjective morphism to a smooth affine curve \(C\). There exists a smooth affine curve \(D\) and a morphism \(D \overset{f'}{\longrightarrow} W\) such that the composed morphism \(D \overset{uf}{\longrightarrow} C\) isn’t a constant map.

Let \(p_1\) be an arbitrary point of \(W\), let \(p\) be its image in \(C\), and let \(F\) be the fibre of the map \(u\) over \(p\). Theorem 5.7.2 shows that there is a map \(D \overset{f'}{\longrightarrow} W\) from a smooth affine curve \(D\) to \(W\) and a point \(q\) of \(D\) such that \(f'(q) = p_1\), and that the image of \(D' = D - \{q\}\) is contained in the complement of \(F\). This is the required map.

\[\square\]
**5.9 Projective Varieties are Proper**

An important property of projective space is that, with its classical topology, it is *compact*, which means that it has these two properties: It is a *Hausdorff space*: Distinct points \( p, q \) of \( X \) have disjoint open neighborhoods, and it is *quasicompact*: If family \( \{ X_i \} \) of open sets covers \( X \), then a finite subfamily covers \( X \).

The next theorem reviews two important properties of compact spaces.

**Theorem.** (i) (Heine-Borel Theorem) A subset of \( \mathbb{R}^n \) is compact if and only if it is closed and bounded.

(ii) Let \( X \xrightarrow{f} Y \) be a continuous map of topological spaces. Suppose that \( X \) is a compact space and that \( Y \) is a Hausdorff space. The image \( f(X) \) is a closed subset of \( Y \), and with the topology induced from \( Y \), the image is compact. □

Let’s use this theorem to verify that \( \mathbb{P}^n \) is compact. The five-dimensional sphere \( S \) of unit length vectors in \( \mathbb{C}^{n+1} \) is bounded, and because it is the zero locus of the equation \( x_0x_0 + \cdots + x_nx_n = 1 \), it is closed. So it is compact. The map \( S \to \mathbb{P}^2 \) that sends a vector \( (x_0, \ldots, x_n) \) to the point of the projective plane with that coordinate vector is continuous and surjective. So \( \mathbb{P}^n \) is compact.

We saw in Section 2.7 that, in the Zariski topology, every variety is a notherian topological space. Consequently, it is quasicompact. But a variety of dimension \( > 0 \) isn’t compact because it isn’t Hausdorff. We show that projective varieties have a property closely related to compactness: They are proper.

Before defining proper varieties, we explain the analogous property of compact spaces.

**Proposition.** Let \( W \) be a closed subset of a product \( Z \times X \), where \( Z \) is a Hausdorff space and \( X \) is a compact space. The image \( Y \) of \( W \) via projection to \( Z \) is a closed subset of \( Z \).

*Proof.* Let \( y_i \) be a sequence of points of \( Y \) that has a limit \( \overline{y} \) in \( Z \). We show that \( \overline{y} \) is a point of \( Y \). For each \( i \), we choose a point \( w_i \) of \( W \) that lies over \( y_i \). The point \( w_i \) is a pair \( (y_i, x_i) \), \( x_i \) being a point of \( X \). Since \( X \) is compact, there is a subsequence of \( x_i \) that has a limit \( \overline{x} \) in \( X \). Passing to subsequences, we may suppose that \( x_i \) has limit \( \overline{x} \). Then \( w_i \) has the limit \( (\overline{y}, \overline{x}) \). Since \( W \) is closed, \( (\overline{y}, \overline{x}) \) is in \( W \), and therefore \( \overline{y} \) is in \( Y \). □

The property of this proposition defines proper varieties.

**Definition.** A variety \( X \) is *proper* if for every variety \( Z \) and every closed subset \( W \) of the product \( Z \times X \), the image \( Y \) of \( W \) via projection to \( Z \) is closed in \( Z \).

Because the image of an irreducible subset is irreducible, the image of a closed subvariety \( Z \) of \( Y \times X \) will be a closed subvariety of \( Y \), if \( X \) is proper.

**Theorem.** Projective varieties are proper.

This theorem is the most important application of the use of curves to characterize closed sets. Before proving it, we give some examples which show how it is used.

**Example.** (singular curves) We assemble the plane curves of a given degree \( d \) into a variety. The number of distinct monomials \( x_0^i x_1^j x_2^k \) of degree \( d = i + j + k \) is the binomial coefficient \( \binom{d + 2}{2} \). We order the monomials arbitrarily, labeling them as \( m_0, \ldots, m_r \), \( r = \binom{d + 2}{2} - 1 \). A homogeneous polynomial of degree \( d \) will be a combination of monomials with complex coefficients \( z_0, \ldots, z_r \), so the homogeneous polynomials of degree \( d \), taken up to scalar factors, are parametrized by a projective space of dimension \( r \) that we denote by \( Z \). Points of \( Z \) correspond bijectively to divisors of degree \( d \) in the projective plane (see Section 1.3.5).

The product space \( Z \times \mathbb{P}^2 \) represents pairs \( (D, p) \), where \( D \) is a divisor of degree \( d \) and \( p \) is a point of \( \mathbb{P}^2 \).

The variable homogeneous polynomial \( f \) may be written as \( f(z, x) \). It is bihomogeneous, linear in \( z \) and of degree \( d \) in \( x \). So the locus \( \Gamma \) of \( \{ f(z, x) = 0 \} \) in \( Z \times \mathbb{P}^2 \) is a (Zariski) closed set whose points are pairs \( (D, p) \) such that \( p \) is a point of the divisor \( D \). The set \( \Sigma \) of pairs \( (D, p) \) such that \( p \) is a singular point of \( D \) is also closed. It is defined by the system of equations \( f_0(z, x) = f_1(z, x) = f_2(z, x) = 0 \), where \( f_i \) is the partial derivative, as usual. The partial derivatives \( f_i \) are bihomogeneous, of degree 1 in \( z \) and degree \( d - 1 \) in \( x \).

The next proposition isn’t easy to prove directly, but the proof becomes easy when one uses the fact that projective space is proper.
**5.9.6. Proposition** The singular divisors of degree \( d \) form a (Zariski) closed subset of the space \( Z \) of all curves of degree \( d \).

*Proof.* Theorem 5.9.4 tells us that the image of the subset \( \Sigma \) via projection to \( Z \) is closed. Its points correspond to singular divisors. \( \square \)

**5.9.7. Example.** (surfaces that contain a line) We go back to the discussion of lines in a surface of Chapter 3. As in that discussion, let \( S \) denote the projective space that parametrizes surfaces of degree \( d \) in \( \mathbb{P}^3 \).

**5.9.8. Proposition** In \( \mathbb{P}^3 \), the surfaces of degree \( d \) that contain a line form a closed subset of the space \( S \).

*Proof.* Let \( G \) be the Grassmanian \( G(2,4) \) of lines in \( \mathbb{P}^3 \), and let \( \Xi \) be the subset of \( G \times S \) of pairs of pairs \([\ell], [S]\) such that \( \ell \subset S \). Lemma 5.3.12 tells us that \( \Xi \) is a closed subset of \( G \times S \). Therefore its image \( W \) in \( S \) is closed. \( \square \)

We now proceed with the proof of Theorem 5.9.4. We will need to tweak the curve criterion for closed sets to prove it. We make use of Corollary 5.8.4 and the next lemma:

**5.9.9. Lemma.** Let \( q \) be a point of a smooth affine curve \( C \), and let \( C' = C - \{q\} \). Every morphism \( C' \xrightarrow{f'} \mathbb{P}^n \) to a projective space extends uniquely to a morphism \( C \xrightarrow{f} \mathbb{P}^n \).

*Proof.* Let \( K \) be the function field of \( C \). The morphism \( f' \) gives us a point of \( \mathbb{P}^n \) with values in \( K \). Such points correspond bijectively, both to morphisms \( C \to \mathbb{P}^n \) and to morphisms \( C' \to \mathbb{P}^n \) (see Corollary 5.4.3). \( \square \)

*Proof of Theorem 5.9.4.* We go back to the notation of Definition 5.9.3. We are given a diagram

\[
\begin{array}{ccc}
W & \xrightarrow{\sigma} & Z \times X \\
\downarrow & & \downarrow \pi \\
Y & \xrightarrow{\tau} & Z
\end{array}
\]

in which \( X \) is a projective variety and \( W \) is a closed subset of \( Z \times X \). The set \( Y \) is the image of \( W \) in \( Z \), and the map \( \sigma \) is the restriction of \( \pi \). We are to prove that \( Y \) is closed in \( Z \). We may assume that \( X \) is a projective space. Also, we know that \( Y \) is constructible. It is a union of locally closed sets \( Y_1, \ldots, Y_k \). It suffices to show that, for \( i = 1, \ldots, k \), the closure \( \overline{Y}_i \) of \( Y_i \) is contained in \( Y \). Let \( \overline{W}_i = \pi^{-1}(\overline{Y}_i) \). This is a closed subset of \( Z \times X \). If \( \overline{W}_i \) maps surjectively to \( \overline{Y}_i \) for each \( i \), then \( W \) maps surjectively to \( Y = \overline{Y}_1 \cup \cdots \cup \overline{Y}_k \), and therefore \( Y = \overline{Y} \). So it suffices to prove the theorem when \( Y \) is open in its closure.

We apply the curve criterion. Suppose given a morphism \( C \xrightarrow{f} X \) from a smooth affine curve to \( Z \) and a point \( q \) of \( C \) such that the image of \( C' = C - \{q\} \) is contained in \( Y \). We must show that \( f(q) \) lies in \( Y \). Corollary 5.8.4 tells us that there is a smooth affine curve \( D \) that fits into a diagram

\[
\begin{array}{ccc}
D & \xrightarrow{f'} & W \\
\downarrow g & & \downarrow \tau \\
C & \xrightarrow{f} & Z
\end{array}
\]

Let \( \tilde{D} \) be the normalization of \( D \) in the function field of \( D \). This smooth affine curve comes with an integral, and therefore surjective, morphism \( \tilde{D} \xrightarrow{\tilde{g}} C \).

We show that the morphism \( f' \) extends to a morphism \( \tilde{D} \xrightarrow{\tilde{f}} W \). Let \( \tilde{D} \xrightarrow{d} Z \) denote the composed morphism \( f\tilde{g} \), and let \( D \xrightarrow{h} X \) be the morphism obtained by restriction from the projection \( Z \times X \to X \). Corollary 5.4.3 shows that \( h \) extends to a morphism \( \tilde{D} \xrightarrow{\tilde{h}} X \). The pair of morphisms \((d, \tilde{h})\) defines a morphism \( \tilde{D} \to Z \times X \) that extends the morphism \( D \to Z \times X \). Since the image of \( D \) is in the closed set \( W \), so is the image of \( \tilde{D} \). This gives us the morphism \( \tilde{D} \xrightarrow{\tilde{f}} W \):
Since the map $\tilde{g}$ is surjective, the image of $C$ is contained in the image $Y$ of $W$. □

5.10 Fibre Dimension

A function $Y \xrightarrow{\delta} Z$ from a variety to the integers is constructible if, for every integer $n$, the set of points of $Y$ such that $\delta(p) = n$ is constructible, and $\delta$ is upper semicontinuous if for every $n$, the set of points such that $\delta(p) \geq n$ is closed. For brevity, we may refer to an upper semicontinuous function as semicontinuous, though the term is ambiguous, since a function might be lower semicontinuous.

If a function $\delta$ on a curve $C$ is semicontinuous, it will be a constant $c$ on a nonempty open subset $U$ and its value on points not in $U$ will be greater or equal to $c$.

The next curve criterion for semicontinuous functions follows from the criterion for closed subvarieties.

5.10.1. Proposition. (curve criterion for semicontinuity) A function $Y \xrightarrow{\delta} Z$ is semicontinuous if and only if it is a constructible function, and for every morphism $C \xrightarrow{f} Y$ from an affine curve $C$ to $Y$, the composition $\delta \circ f$ is a semicontinuous function on $C$.

Let $Z$ be a closed subset of a variety $X$, and let $p$ be a point of $Z$. The local dimension of $Z$ at $p$ is the maximum dimension among the irreducible components of $Z$ that contain $p$. For example, in $\mathbb{P}^1$, let $L$ be a line that meets a plane $H$ at a point $p$, and let $Z = H \cup L$. The local dimension of $Z$ at every point of $H$ is 2, and is 1 at points of $L$ different from $p$.

Let $Y \xrightarrow{f} X$ be a morphism, let $q$ be a point of $Y$, and let $F$ be the fibre of $f$ over $p = f(q)$. The fibre dimension $\delta(q)$ of $f$ at $q$ is the local dimension of the fibre $F$ at $q$.

5.10.2. Theorem. (semicontinuity of fibre dimension) Let $Y \xrightarrow{u} X$ be a morphism of varieties, and let $\delta(q)$ denote the fibre dimension at a point $q$ of $Y$.

(i) Suppose that $X$ is a smooth curve, that $Y$ has dimension $n$, and that $u$ is not a constant map from $Y$ to a point of $X$. Then $\delta$ is the constant function $n - 1$: Every fibre has constant dimension equal to $n - 1$.

(ii) Suppose that the image of $Y$ contains a nonempty open subset of $X$, and let the dimensions of $X$ and $Y$ be $m$ and $n$, respectively. There is a nonempty open subset $X'$ of $X$ such that $\delta(q) = n - m$ for every point $q$ in the inverse image of $X'$.

(iii) $\delta$ is a semicontinuous function on $Y$.

The proof of this theorem makes a good exercise.