Problem 1:

For $d \equiv 1$ modulo 8, 2 is not prime. Indeed:

$$2 \mid \left( \frac{1 + \sqrt{d}}{2} \right) \left( \frac{1 - \sqrt{d}}{2} \right)$$

but 2 does not divide either of the two factors. Meanwhile, for $d \equiv 5$ modulo 8, we will show that 2 is indeed prime. Assume that:

$$2 \mid \left( \frac{x + y\sqrt{d}}{2} \right) \left( \frac{x' + y'\sqrt{d}}{2} \right) \quad (1)$$

where $x, y$ (respectively $x', y'$) are both even or both odd, and assume that neither of the two factors is divisible by 2. This implies that either $x, y$ (respectively $x', y'$) are both odd, or one is congruent to 0 mod 4, while the other one is congruent to 2 mod 4. In either of these cases, we have:

$$(x^2, y^2) \equiv (1, 1) \text{ or } (0, 4) \text{ or } (4, 0) \mod 8$$

and therefore:

$$x^2 - y^2d \equiv 4 \mod 8 \implies N\left( \frac{x + y\sqrt{d}}{2} \right) \equiv 1 \mod 2$$

Together with the analogous result for $x', y'$, this implies that the factors in the right hand side of (1) both have odd norm. Since 2 has even norm, this contradicts (1).

Problem 2:

a) The lattices in question are actually maximal ideals, because the quotients $R/P$ and $R/Q$ are fields. To see this, note that as a set, $|R/P| = 2$ while $|R/Q| = 3$. It is easy to see that all rings of cardinality 2 are isomorphic to $\mathbb{F}_2$ (since their only elements are 0 and 1) and all rings of cardinality
3 are isomorphic to \( \mathbb{F}_3 \) (since their only elements are 0, 1 and \(-1\)).

b) We claim that (6) = \( P\bar{P}Q\bar{Q} \). To see this, note that:

\[ P\bar{P} = (2, \delta)(2, -\delta) = (4, 2\delta, -\delta^2) = (4, 2\delta, 6) = (2) \]
\[ Q\bar{Q} = (3, \delta)(3, -\delta) = (9, 3\delta, -\delta^2) = (9, 3\delta, 6) = (3) \]

c) In the case at hand, we have \( \mu = \sqrt{8} \). Theorem 13.7.10 (b) says that the class group is generated by prime ideals of norm 2. The only such ideals are \( P \) and \( \bar{P} \), because the existence of any other prime ideal of norm 2 would contradict the unique factorization (2) = \( P\bar{P} \). Since:

\[ \langle P \rangle + \langle P \rangle = \langle (2) \rangle = 0 \]

in the class group, we conclude that the class group is \( \mathbb{Z}_2 \).

**Problem 3:**

We will refine the standard argument. Consider the operator of multiplication by a non-zero element \( r \in R \):

\[ R \xrightarrow{f} R, \quad f(\alpha) = r \cdot \alpha \]

This is a linear transformation of \( F \)-vector spaces, since \( f(\alpha + \beta) = f(\alpha) + f(\beta) \) and \( f(\alpha \lambda) = \lambda f(\alpha) \) for all \( \lambda \in F \). But \( f \) is injective because \( R \) is an integral domain. Since \( R \) is finite dimensional over \( F \), this implies that \( f \) is surjective, hence:

\[ \exists \alpha \in R \text{ such that } 1 = f(\alpha) = r \cdot \alpha \]

This shows that every element of \( R \) has a multiplicative inverse, hence \( R \) is a field.

**Problem 4:**

Since \( \beta \) satisfies the equation \( \beta^3 = 2 \), we have:

\[ K = \mathbb{Q}(\beta) = \left\{ s + \beta t + \beta^2 u, \ s, t, u \in \mathbb{Q} \right\} \]
In other words, we claim that the inverse of any element of the form (2) is still of that form. The reason is that:

$$\frac{1}{s + \beta t + \beta^2 u} = s' + \beta t' + \beta^2 u'$$

for some rational numbers $s', t', u'$. Therefore, let us write:

$$x_i = s_i + \beta t_i + \beta^2 u_i$$

for rational numbers $s_i, t_i, u_i$. Squaring the above formula gives us:

$$x_i^2 = (s_i^2 + 2t_iu_i + \beta(2u_i^2 + 2s_it_i) + \beta^2(t_i^2 + 2u_is_i))$$

The equation $x_1^2 + \ldots + x_k^2 = -1$ would imply:

$$\sum_{i=1}^{k}(s_i^2 + 2t_iu_i) = -1, \quad \sum_{i=1}^{k}(u_i^2 + s_it_i) = 0, \quad \sum_{i=1}^{k}(t_i^2 + 2u_is_i) = 0$$

Multiply the middle equation by 2 and add all of them together:

$$\sum_{i=1}^{k}(s_i^2 + 2u_i^2 + t_i^2 + 2t_iu_i + 2s_it_i + 2u_is_i) = -1$$

We obtain a contradiction, since the left hand side is $\sum_{i=1}^{k}[(s_i + t_i + u_i)^2 + u_i^2]$

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1 To see this, multiply the two sides, replace all $\beta^3$ by 2, set the coefficients of $1, \beta, \beta^2$ equal to 1, 0, 0, and use the resulting equations to solve for $s', t', u'$ in terms of $s, t, u$