Notes for 02/24: Symmetric polynomials

In Redmond McNamara’s notes, he constructs the representations:

\[
\{ \text{span of polytabloids} \} = S^\lambda \subset M^\lambda = \{ \text{span of tabloids} \}
\]

of the symmetric group \(S_n\), where \(\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots)\) goes over all partitions of the natural number \(n\). He shows that the \(\{S^\lambda\}_{\lambda \text{ partition}}\) form a complete collection of the irreducible representations of the symmetric group. Moreover, while not explicit, his argument also shows that we have a decomposition:

\[
M^\lambda \cong \bigoplus_{\mu \triangleright \lambda} K^\lambda_{\mu} \cdot S^\mu
\]

(1)

where \(\triangleright\) denotes the dominance ordering on partitions:

\[
\mu \triangleright \lambda \iff \mu_1 + \ldots + \mu_i \geq \lambda_1 + \ldots + \lambda_i \quad \forall i
\]

The natural numbers \(K^\lambda_{\mu}\) are called Kostka numbers and they are important enough to warrant their own Wikipedia page. We will not be studying them, but what I’d like to take from formula (1) is the fact that the representations \(M^\lambda\) are triangular in terms of the representations \(S^\lambda\). Therefore, their characters are also triangular in terms of each other:

\[
\chi_{M^\lambda}(C_\delta) = \chi_{S^\lambda}(C_\delta) + \sum_{\mu \triangleright \lambda} K^\lambda_{\mu} \cdot \chi_{S^\mu}(C_\delta)
\]

(2)

for any partition \(\delta = (\delta_1 \geq \delta_2 \geq \ldots \geq \delta_s)\), where \(C_\delta \subset S_n\) denotes the conjugacy class consisting of permutations with cycle type \(\delta\).

**Proposition:** The character of the representation \(M^\lambda\) is given by:

\[
p_\delta(x_1, \ldots, x_N) = \sum_{\lambda \text{ is a partition of } n} m_\lambda(x_1, \ldots, x_N) \cdot \chi_{M^\lambda}(C_\delta)
\]

(3)

for any \(N \geq n\), where the power sum functions are defined by:

\[
p_k(x_1, \ldots, x_N) = x_1^k + \ldots + x_N^k \quad \text{and} \quad p_\delta = p_{\delta_1} p_{\delta_2} \ldots p_{\delta_s}
\]

and the monomial symmetric functions are defined by:

\[
m_\lambda(x_1, \ldots, x_N) = \text{symmetrization of } x_1^{\lambda_1} \ldots x_N^{\lambda_N}
\]

(4)
where we set $\lambda_k = 0$ for $k >$ the number of parts of $\lambda$. By “symmetrization” we mean summing over all permutations of the variables $x_i$ in such a way that $m_\lambda$ becomes a symmetric polynomial. For example, we have:

$$m_{(1,1,1)}(x_1, \ldots, x_N) = \sum_{1 \leq i < j < k \leq N} x_i x_j x_k$$

$$m_{(2,1)}(x_1, \ldots, x_N) = \sum_{1 \leq i \neq j \leq N} x_i^2 x_j$$

$$m_{(3)}(x_1, \ldots, x_N) = \sum_{1 \leq i \leq N} x_i^3 = p_3$$

**Proof:** Recall that $M^\lambda$ has a basis given by tabloids, which are just fillings of the boxes in the Young diagram $\lambda$ with the numbers $1, \ldots, n$. Two tabloids are considered identical if they can be obtained from each other by permuting the rows. A permutation $\sigma$ acts on such a tabloid by, well, permuting the numbers. Therefore, $\sigma$ preserves a certain tabloid if and only if the rows of the tabloid are full cycles of $\sigma$, so we conclude that:

$$\text{Tr}_{M^\lambda}(\sigma) = \# \text{ of ways to fill the rows of } \lambda \text{ with full cycles of } \sigma$$

Let us assume that $\sigma \in C_\delta$, where $\delta$ consists of $\alpha^i$ cycles of length $i$, for all $i$. Then the above equality yields:

$$\chi_{M^\lambda}(C_\delta) = \prod_{i} \frac{\alpha^i}{\beta^i_1 \beta^i_2 \cdots}$$

In other words, the sum goes over all ways to divide the cycles of $\sigma \in C_\delta$ among the rows of $\lambda$. The ratio of factorials (multinomial coefficient) is there because we have the freedom to choose in which row of $\lambda$ we place each of the $\alpha^i$ cycles of length $i$. Then multiply the above formula by the symmetric polynomial (4), and sum over $\lambda$. We get:

$$\sum_{\text{partition of } n} m_\lambda(x_1, \ldots, x_N) \cdot \chi_{M^\lambda}(C_\delta) = \sum_{\alpha^i = \beta^i_1 + \beta^i_2 + \cdots} \prod_{i} \frac{\alpha^i}{\beta^i_1 \beta^i_2 \cdots} \prod_{j} x_j^{1 \cdot \beta^i_1 + 2 \cdot \beta^i_2 + \cdots} =$$

$$= \prod_{i} \left( \sum_{\alpha^i = \beta^i_1 + \beta^i_2 + \cdots} x_1^{\beta^i_1} x_2^{\beta^i_2} \cdots x_N^{\beta^i_N} \cdot \frac{\alpha^i}{\beta^i_1 \beta^i_2 \cdots} \right) = \prod_{i} (x_1^{\alpha^i} + x_2^{\alpha^i} + \cdots + x_N^{\alpha^i})^{\alpha^i}$$
The right hand side is precisely $p_{\delta}(x_1, \ldots, x_N)$, as required. Note that the equality in the last formula is the multinomial theorem (see e.g. Wikipedia).

Combining (2) and (3) gives us:

$$p_{\delta}(x_1, \ldots, x_N) = \sum_{\mu \text{ partition of } n} s_{\mu}(x_1, \ldots, x_N) \cdot \chi_{S^n}(C_{\delta}) \quad (5)$$

where the functions $s_{\mu}$ are defined by:

$$s_{\mu}(x_1, \ldots, x_N) = \sum_{\lambda \subseteq \mu} K_{\mu}^\lambda \cdot m_\lambda(x_1, \ldots, x_N) \quad \forall \text{ partition } \mu \quad (6)$$

The $s_{\mu}$ are called **Schur polynomials** simply because they are symmetric polynomials in the variables $x_i$. In fact, the above formula shows that they have positive integer coefficients, but these coefficients are as mysterious as the Kostka numbers themselves. To get a grasp on this, let’s transform (5) by multiplying both sides with $\chi_{S^n}(C_{\lambda})$ for any partition $\lambda$:

$$p_{\delta}(x_1, \ldots, x_N) \cdot \chi_{S^n}(C_{\lambda}) = \sum_{\mu \text{ partition of } n} s_{\mu}(x_1, \ldots, x_N) \cdot \chi_{S^n}(C_{\delta}) \chi_{S^n}(C_{\lambda})$$

Let’s multiply both sides of the above equality by the cardinality of the cycle class $C_{\delta}$ divided by the order of $S_n$, which you can easily check equals:

$$\frac{|C_{\delta}|}{n!} = \frac{1}{z_{\delta}} \quad \text{where} \quad z_{\delta} = 1^{\alpha_1} \alpha_1! 2^{\alpha_2} \alpha_2! \ldots$$

if the cycle type $\delta$ contains $\alpha_1$ cycles of length 1, $\alpha_2$ cycles of length 2 etc.

$$\frac{p_{\delta}(x_1, \ldots, x_N)}{z_{\delta}} \cdot \chi_{S^n}(C_{\delta}) = \sum_{\mu \text{ partition of } n} s_{\mu}(x_1, \ldots, x_N) \cdot \sum_{\sigma \in C_{\delta}} \chi_{S^n}(\sigma) \chi_{S^n}(\sigma) \frac{n!}{n!}$$

Sum the above formula over all cycle types $\delta$:

$$\sum_{\text{cycle type } \delta} \frac{p_{\delta}(x_1, \ldots, x_N)}{z_{\delta}} \cdot \chi_{S^n}(C_{\delta}) = \sum_{\mu \text{ partition of } n} s_{\mu}(x_1, \ldots, x_N) \cdot \sum_{\sigma \in S_n} \chi_{S^n}(\sigma) \chi_{S^n}(\sigma) \frac{n!}{n!}$$
You see where this is heading. Because the representations $S^\lambda$ are all irreducible, orthogonality of characters tells us that the right hand side is non-zero only of $\lambda = \mu$. We conclude that:

$$s_\lambda(x_1, \ldots, x_N) = \sum_{\delta \text{ is a cycle type}} \frac{p_\delta(x_1, \ldots, x_N)}{z_\delta} \cdot \chi_{S^\lambda}(C_\delta)$$

(7)

**Definition 1:** Given any representation $V$ of $S_n$, define its *Frobenius character* as the symmetric polynomial:

$$F_V(x_1, \ldots, x_N) = \sum_{\delta \text{ is a cycle type}} \frac{p_\delta(x_1, \ldots, x_N)}{z_\delta} \cdot \chi_V(C_\delta)$$

(8)

This symmetric polynomial contains the full information about the character of the representation $V$. Indeed, it’s enough to expand $F_V$ in terms of the power sum functions $p_\delta$ and the coefficients will tell you all the values of the character $\chi_V$. So we allow characters, which were originally functions $S_n \to \mathbb{C}$, to be replaced by symmetric polynomials:

$$\chi_V \rightsquigarrow F_V$$

(9)

Formula (7) says precisely that the Frobenius character of the irreducible representations are the Schur polynomials. But how does this viewpoint help to compute the Schur polynomials? The answer is to use (9) in order to replicate the Hermitian inner product of characters.

**Definition 2:** Consider the *Hall inner product* on the space of symmetric polynomials, given by the formula:

$$\langle p_\delta, p_\epsilon \rangle_{\text{Hall}} = \begin{cases} z_\delta & \text{if } \delta = \epsilon \\ 0 & \text{otherwise} \end{cases}$$

It is easy to see from the definition (8) that:

$$\langle F_V, F_{V'} \rangle_{\text{Hall}} = \langle \chi_V, \chi_{V'} \rangle$$

Since the characters $\chi_{S^\lambda}$ are orthogonal in virtue of $S^\lambda$ being irreducible representations, we infer that the Schur polynomials are also orthogonal with
respect to the Hall inner product. Hence we obtain the following characterization of Schur polynomials:

**Theorem:** The Schurs $s_{\lambda}$ are the unique collection of symmetric polynomials which are orthogonal with respect to the Hall inner product, and are upper triangular in terms of monomial symmetric functions, see (6).

Note that this statement is just Gram-Schmidt orthogonalization: given a vector space with an inner product and a basis $e_1, \ldots, e_n$, there is a unique orthogonal basis which is upper triangular in the $e_i$'s. It turns out that this description of Schur polynomials actually makes them very easy to compute. Here are a few examples:

\[
s_{(1,\ldots,1)}(x_1, \ldots, x_N) = \sum_{1 \leq i_1 < i_2 < \ldots < i_n \leq N} x_{i_1} \ldots x_{i_n}
\]

\[
s_{(n)}(x_1, \ldots, x_N) = \sum_{1 \leq i_1 \leq i_2 \leq \ldots \leq i_n \leq N} x_{i_1} \ldots x_{i_n}
\]

and some special cases:

\[
s_{(2,1)}(x_1, \ldots, x_N) = m_{(2,1)}(x_1, \ldots, x_N) + m_{(1,1,1)}(x_1, \ldots, x_N)
\]

\[
s_{(2,2)}(x_1, \ldots, x_N) = m_{(2,2)}(x_1, \ldots, x_N) + m_{(2,1,1)}(x_1, \ldots, x_N)
\]

\[
s_{(3,1)}(x_1, \ldots, x_N) = m_{(3,1)}(x_1, \ldots, x_N) + 2m_{(2,1,1)}(x_1, \ldots, x_N) + 3m_{(1,1,1,1)}(x_1, \ldots, x_N)
\]

There are a lot of fascinating results in combinatorics and rep theory about Schur polynomials, such as the Littlewood-Richardson rule, the Giambelli and Jacobi-Trudi formulas, the Cauchy formula etc. Check them out!