Suppose we have a finite group and a subgroup:
\[ H \subset G \]
then we construct two “functions”:
\[ \{ \text{reps of } G \} \xrightarrow{\text{Res}_H^G} \{ \text{reps of } H \} \xrightarrow{\text{Ind}_H^G} \{ \text{reps of } G \} \]

The first arrow, restriction, is the easiest. Given a representation \( G \rhd V \), we define its restriction as the same vector space, with the same action:
\[ \text{Res}_H^G V = V \quad \text{with} \quad H \rhd V \text{ deduced from the action of } G \supset H \]

It is clear that if \( K \subset H \subset G \) are three groups, then:
\[ \text{Res}_H^K (\text{Res}_H^G V) = \text{Res}_K^G V \quad (1) \]

Induction is a bit more involved, since it is given by a bigger space:
\[ \text{Ind}_H^G W = \left\{ \text{functions } f : G \to W \text{ s.t. } f(hg) = h \cdot f(g) \text{ for all } h \in H, g \in G \right\} \]

In other works, points in the above vector space are functions from a finite set into \( V \), which satisfy a certain property. It is clear that such functions can be added together and multiplied by a scalar. The action:
\[ G \rhd \text{Ind}_H^G W \quad \text{is defined by} \]
\[ x \in G \quad \text{ sends } \quad G \xrightarrow{f(g)} W \quad \text{ to } \quad G \xrightarrow{f(gx)} W \]

To describe a basis of the induced representation, consider a collection of right cosets:
\[ G = \bigsqcup_{i=1}^{r} Hx_i \quad \text{ where } r = \frac{|G|}{|H|} \]

and a basis \( \{e_1, ..., e_d\} \) of \( W \). Then a basis of \( \text{Ind}_H^G W \) is given by the functions:
\[ G \xrightarrow{f_i^j} W \quad f_i^j(hx_\alpha) = \begin{cases} h \cdot e_j & \text{if } \alpha = i \\ 0 & \text{otherwise} \end{cases} \quad \forall i \in \{1, ..., r\}, \ j \in \{1, ..., d\} \]
**Theorem (Frobenius reciprocity):** For any representations $G \curvearrowright V$ and $H \curvearrowright H$, their characters satisfy:

$$\langle \chi_V, \chi_{\text{Ind}^G_H W} \rangle = \langle \chi_{\text{Res}^G_H V}, \chi_W \rangle \quad (2)$$

**Sketch of proof:** As we saw last week, it is enough to show that:

$$\text{Hom}_G (V, \text{Ind}^G_H W) \cong \text{Hom}_H (\text{Res}^G_H V, W)$$

where $\text{Hom}_G$ refers to the vector space of $G$–invariant linear transformations. The isomorphism between the above Hom sets is given by:

$$V \xrightarrow{\Phi} \text{Ind}^G_H W \quad \rightsquigarrow \quad \text{Res}^G_H V \xrightarrow{\Psi} W, \quad \Psi(v) = \Phi(v)(1)$$

$$\text{Res}^G_H V \xrightarrow{\Psi} W \quad \rightsquigarrow \quad V \xrightarrow{\Phi} \text{Ind}^G_H W, \quad \Phi(v)(g) = \Psi(g \cdot v) \quad \blacksquare$$

It’s an easy exercise to show that (1) and (2) imply that:

$$\text{Ind}^G_H (\text{Ind}^H_K W) = \text{Ind}^G_K W \quad (3)$$