Exercise 6  In class we showed that, in the nearest neighbor random walk on $\mathbb{Z}$, $\{X_n\}_{n \geq 1}$, the time $T_0$ of first return to 0 has the following probability distribution:

$$\mathbb{P}[T_0 = n] = \frac{2}{n-1} \binom{n-1}{n/2} p^{n/2} q^{n/2}.$$ 

Prove, by a direct computation, that

$$\mathbb{E}[s^{T_0}] = 1 - \sqrt{1 - 4s^2pq}.$$ 

We first remark that the value given in the problem statement for $\mathbb{P}[T_0 = n]$ has a caveat: since we have to take an even number of steps to get back to 0, and it is the time of first return, we have specifically that $\mathbb{P}[T_0 = n] = \frac{2}{n-1} \binom{n-1}{n/2} p^{n/2} q^{n/2}$ for $n = 2k$ for $k = 1, 2, 3, \ldots$, and $\mathbb{P}[T_0 = n] = 0$ otherwise. Then, we can write

$$\mathbb{E}[s^{T_0}] = \sum_{k=1}^{\infty} \left( s^{2k} \cdot \mathbb{P}[T_0 = 2k] \right).$$

To proceed, we make the following claim:

Claim.  \( \frac{1}{2k-1} \binom{2k-1}{k} = 2(-4)^{k-1} \binom{1/2}{k} \).

Proof.  We write

$$2 \binom{1/2}{k} = 2 \frac{(1/2)(-1/2)(-3/2) \cdots ((2k-3)/2)}{k!} = \frac{(-1)^{k-1}}{k!} \left( \frac{1}{2} \cdot \frac{3}{2} \cdots \frac{2k-3}{2} \right)$$

$$= \frac{(-1)^{k-1}}{k! \cdot 2^{k-1}} (1 \cdot 3 \cdots (2k-3)) = \frac{1}{k! \cdot (-2)^{k-1}} (1 \cdot 3 \cdots (2k-3)).$$

Then,

$$2(-4)^{k-1} \binom{1/2}{k} = \frac{(-4)^{k-1}}{k! \cdot (-2)^{k-1}} (1 \cdot 3 \cdots (2k-3)) = \frac{2^{k-1}}{k!} (1 \cdot 3 \cdots (2k-3))$$

$$= \frac{2^{k-1}}{k!} (1 \cdot 3 \cdots (2k-3)) \cdot \frac{2 \cdot 4 \cdot 6 \cdots (2k-2)}{2 \cdot 4 \cdot 6 \cdots (2k-2)}$$

$$= \frac{2^{k-1}}{k!} (1 \cdot 3 \cdots (2k-3)) \cdot \frac{2 \cdot 4 \cdot 6 \cdots (2k-2)}{2^{k-1} \cdot 1 \cdot 2 \cdot 3 \cdots (k-1)} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdots (2k-2)}{k!(k-1)!}.$$
\[\frac{1 \cdot 2 \cdot 3 \cdot 4 \cdots (2k - 2)}{k!(k-1)!} \cdot \frac{2k-1}{2k-1} = \frac{1}{2k-1} \cdot \frac{(2k-1)!}{k!(k-1)!} = \frac{1}{2k-1} \binom{2k-1}{k},\]
as desired. Thus, the claim is proven.

Now, using our claim, we write

\[\mathbb{P}[T_0 = 2k] = \frac{2}{2k-1} \binom{2k-1}{k} p^k q^k = 2 \cdot (-4)^{-k-1} \binom{1/2}{k} p^k q^k = -(-4)^k \binom{1/2}{k} p^k q^k.\]

Then,

\[\mathbb{E}[s^{T_0}] = \sum_{k=1}^{\infty} \left( s^{2k} \cdot \mathbb{P}[T_0 = 2k] \right) = \sum_{k=1}^{\infty} \left( -s^{2k} \cdot (-4)^k \binom{1/2}{k} p^k q^k \right) = \sum_{k=1}^{\infty} \left( \binom{1/2}{k} (-4s^2 pq)^k \right).\]

By the Binomial Theorem, \(\sum_{k=0}^{\infty} \binom{a}{k} x^k = (1 + x)^a\) for \(|x| < 1\), so we know that

\[\sum_{k=0}^{\infty} \left( \binom{1/2}{k} (-4s^2 pq)^k \right) = \sqrt{1 - 4s^2 pq},\]

and

\[\sum_{k=0}^{\infty} \left( \binom{1/2}{k} (-4s^2 pq)^k \right) = 1 + \sum_{k=1}^{\infty} \left( \binom{1/2}{k} (-4s^2 pq)^k \right)\]

so

\[\sum_{k=1}^{\infty} \left( \binom{1/2}{k} (-4s^2 pq)^k \right) = \sqrt{1 - 4s^2 pq} - 1\]

. Thus,

\[\mathbb{E}[s^{T_0}] = -\sum_{k=1}^{\infty} \left( \binom{1/2}{k} (-4s^2 pq)^k \right) = -\left( \sqrt{1 - 4s^2 pq} - 1 \right) = 1 - \sqrt{1 - 4s^2 pq}.
\]
Exercise 7  The Smiths receive the paper every morning and place it on a pile after reading it. Each afternoon, with probability $1/3$, someone takes all the papers in the pile and puts them in the recycling bin. Also, if there are 5 papers in the pile, Mr. Smith (with probability 1) takes the papers to the bin. Consider the number of papers $X_n$ in the pile in the evening of day $n$. Is it reasonable to model this by a Markov chain? If so, what are the state space and the transition matrix?

Yes, it is reasonable to model this by a Markov chain. Specifically we use the state space $\{0, 1, 2, 3, 4\}$, since there cannot be 5 papers at the end of day $n$ because if there were 5 papers in the morning, Mr. Smith would have taken all of them to the bin during the afternoon. Let $X_k$ be the state at the end of day $n$, and let $X_k \neq 4$. Then, there are two possibilities for $X_{k+1}$; specifically, there is a $1/3$ chance that $X_{k+1} = 0$, and there is a $2/3$ chance that $X_{k+1} = X_k + 1$. The probability of $X_{k+1}$ being anything else is 0. In the unique case of $X_k = 4$, we note that $X_{k+1}$ must be 0 (with probability 1) since if a fifth paper was added, Mr. Smith would automatically throw them all out. Using this, we can then construct the probability transition matrix as follows:

$$P = \begin{bmatrix}
\frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\
\frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 \\
\frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 \\
\frac{1}{3} & 0 & 0 & 0 & \frac{2}{3} \\
1 & 0 & 0 & 0 & 0
\end{bmatrix}.$$

Exercise 8  Consider a Markov chain with state space $\{0, 1\}$ and transition matrix

$$P = \begin{bmatrix}
\frac{1}{3} & \frac{2}{3} \\
\frac{3}{4} & \frac{1}{4}
\end{bmatrix}.$$

Assuming that the chain starts in state 0 at time $n = 0$, what is the probability that it is in state 1 at time $n = 2$?

We have that

$$P[X_2 = 1|X_0 = 0] = P[X_1 = 0, X_2 = 1|X_0 = 0] + P[X_1 = 1, X_2 = 1|X_0 = 0]$$

which by the Markov property is precisely

$$P[X_1 = 0|X_0 = 0] \cdot P[X_2 = 1|X_1 = 0] + P[X_1 = 1|X_0 = 0] \cdot P[X_2 = 1|X_1 = 1] = \frac{1}{3} \cdot \frac{2}{3} + \frac{2}{3} \cdot \frac{1}{4} = \frac{2}{9} + \frac{2}{12} = \frac{7}{18}.$$
Exercise 9 (K&T 1.5 p.99)  A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix (for the states $\{0, 1, 2\}$):

\[
P = \begin{bmatrix}
0.3 & 0.2 & 0.5 \\
0.5 & 0.1 & 0.4 \\
0.5 & 0.2 & 0.3
\end{bmatrix}
\]

and initial distribution: $p_0 = 0.5$, $p_1 = 0.5$, $p_2 = 0$. Determine the probabilities:

1. $P[X_0 = 1, X_1 = 1, X_2 = 0]$,
2. $P[X_0 = 1, X_1 = 1, X_3 = 0]$.

Using the Markov property, we have that

\[
P[X_0 = 1, X_1 = 1, X_2 = 0] = P[X_2 = 0|X_1 = 1] \cdot P[X_1 = 1|X_0 = 1] \cdot P[X_0 = 1]
= P_{10} \cdot P_{11} \cdot p_0 = 0.5 \cdot 0.1 \cdot 0.5 = 0.025.
\]

Using the Markov property, we have that

\[
P[X_0 = 1, X_1 = 1, X_3 = 0]
= P[X_3 = 0|X_2 = 0] \cdot P[X_2 = 0|X_1 = 1] \cdot P[X_1 = 1|X_0 = 1] \cdot P[X_0 = 1]
+ P[X_3 = 0|X_2 = 1] \cdot P[X_2 = 1|X_1 = 1] \cdot P[X_1 = 1|X_0 = 1] \cdot P[X_0 = 1]
+ P[X_3 = 0|X_2 = 2] \cdot P[X_2 = 2|X_1 = 1] \cdot P[X_1 = 1|X_0 = 1] \cdot P[X_0 = 1]
= P_{00} \cdot P_{10} \cdot P_{11} \cdot p_0 + P_{10} \cdot P_{11} \cdot P_{11} \cdot p_0 + P_{20} \cdot P_{12} \cdot P_{11} \cdot p_0
= p_0 \cdot P_{11} \cdot (P_{00} \cdot P_{10} + P_{10} \cdot P_{11} + P_{20} \cdot P_{12})
= 0.5 \cdot 0.1 \cdot (0.3 \cdot 0.5 + 0.5 \cdot 0.1 + 0.5 \cdot 0.4)
= 0.5 \cdot 0.1 \cdot (0.15 + 0.05 + 0.2) = 0.5 \cdot 0.1 \cdot 0.4 = 0.02.
\]
Exercise 10 (K&T 1.4 p.100)  The random variables $\xi_1, \xi_2, \ldots$ are independent identically distributed, with common probability distribution

$$
P[\xi = 0] = 0.1, \quad P[\xi = 1] = 0.3, \quad P[\xi = 2] = 0.2, \quad P[\xi = 3] = 0.4.
$$

Set $X_0 = 0$ and $X_n = \max (\xi_1, \ldots, \xi_n)$ be the largest $\xi$ observed to date. Determine the transition probability matrix for the Markov chain $\{X_n\}$.

We first note that $X_{k+1} \geq X_k$ for all $k$, since if $\xi_{k+1} \leq X_k$, $X_{k+1} = X_k$, and if $\xi_{k+1} > X_k$, then $X_{k+1} = \xi_{k+1} > X_k$. Thus, $P_{ab} = 0$ if $a > b$, so

$$
P_{10} = P_{20} = P_{21} = P_{30} = P_{31} = P_{32} = 0.
$$

In the cases where $a > b$, we have simply that

$$
P[X_{k+1} = a | X_k = b] = P[\xi = a],
$$

so

$$
P_{01} = 0.3 \quad P_{02} = P_{12} = 0.2 \quad P_{03} = P_{13} = P_{23} = 0.4.
$$

Finally, we note that

$$
P[X_{k+1} = a | X_k = a] = \sum_{j=0}^{a} P[\xi = j]
$$

so

$$
P_{00} = P[\xi = 0] = 0.1
$$

$$
P_{11} = P[\xi = 0] + P[\xi = 1] = 0.1 + 0.3 = 0.4
$$

$$
P_{22} = P[\xi = 0] + P[\xi = 1] + P[\xi = 2] = 0.1 + 0.3 + 0.2 = 0.6
$$

$$
P_{33} = P[\xi = 0] + P[\xi = 1] + P[\xi = 2] + P[\xi = 3] = 0.1 + 0.3 + 0.2 + 0.4 = 1
$$

Thus, we can construct the transition probability matrix

$$
P = \begin{bmatrix}
0.1 & 0.3 & 0.2 & 0.4 \\
0.0 & 0.4 & 0.2 & 0.4 \\
0.0 & 0.0 & 0.6 & 0.4 \\
0.0 & 0.0 & 0.0 & 1.0 \\
\end{bmatrix}.
$$