Problem 1: Let $N_t$ be a birth/death process with: \( \lambda_n = 4 \) for \( n \neq 8 \), \( \lambda_8 = 0 \), \( \mu_n = 5 \) for \( n \neq 9 \) and \( \mu_9 = 0 \).

\( a \) Describe the communicating classes.

\( b \) Assuming \( N_0 \leq 8 \), is the process recurrent or transient? In the former case find, if it exists, the equilibrium distribution. In the latter case, find if explosion occurs with positive probability.

\( c \) Assuming \( N_0 \geq 9 \), is the process recurrent or transient? In the former case find, if it exists, the equilibrium distribution. In the latter case, find if explosion occurs with positive probability.

Solution: (a) There are two communicating classes: \( \{0, 1, \ldots, 8\} \) and \( \{9, 10, 11, \ldots\} \).

\( b \) Since the class \( \{0, 1, \ldots, 8\} \) is finite, if \( N_0 \leq 8 \) the process becomes an irreducible continuous time Markov chain on a finite state space, with infinitesimal generator matrix

\[
A = \begin{pmatrix}
-4 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & -9 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 5 & -9 & 4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 5 & -9 & 4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 5 & -9 & 4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 5 & -9 & 4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 5 & -9 & 4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 5 & -9 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -5 \\
\end{pmatrix}
\]

Hence, by the Perron-Frobenious Theorem, it is recurrent and it admits a (unique) equilibrium limiting distribution \( \bar{\pi} = (x_0, x_1, \ldots, x_8) \), solution of \( \bar{\pi}A = 0 \). Solving the system of equations we get:

\[
x_8 = \frac{4}{5} x_7, \quad x_7 = \frac{4}{5} x_6, \ldots, x_1 = \frac{4}{5} x_0,
\]

namely

\[
x_n = \left(\frac{4}{5}\right)^n x_0, \quad n = 0, \ldots, 8,
\]

and imposing the normalization condition \( x_0 + \cdots + x_9 = 1 \), we get

\[
x_0 = \frac{1}{\sum_{n=0}^{8} \left(\frac{4}{5}\right)^n} = \frac{5^8}{5^9 - 4^9}.
\]

Hence,

\[
x_n = \frac{4^n 5^8 - n}{5^9 - 4^9}, \quad n = 0, \ldots, 8.
\]

\( c \) For \( N_0 \geq 9 \), the process is the same as a MM1 queueing model with \( \lambda_n = 4 \), \( \mu_n = 5 \) for every \( n \), shifted (by 9). Since the series \( \sum_{n=0}^{\infty} \frac{\lambda_n}{\mu_n+\mu_{n+9}} = \sum_{n=0}^{\infty} \left(\frac{4}{5}\right)^n \) is divergent, we know that the process is recurrent. In fact, since the series \( \sum_{n=0}^{\infty} \frac{\lambda_n}{\mu_n+\mu_{n+9}} = \sum_{n=0}^{\infty} \left(\frac{4}{5}\right)^n \) is convergent, the process is positive recurrent, and it admits a unique limiting equilibrium distribution \( \bar{\pi} \). It is given by

\[
\bar{\pi}_{n+9} = \frac{\lambda_n / \mu_n 5^9 - \lambda_9 / \mu_9}{\sum_{m=0}^{\infty} \frac{\lambda_m}{\mu_m + \mu_{m+9}} 5^m} = \frac{\left(\frac{4}{5}\right)^n}{\sum_{m=0}^{\infty} \left(\frac{4}{5}\right)^m} = \frac{4^n}{\frac{5^n+1}{5^{n+1}}}
\]

Problem 2: A small barber shop, operated by a single barber, has room for only two costumers. Potential costumers arrive at a Poisson rate of 3 per hour, and the successive serving times are independent exponential random variables of mean 1/4 hour.

\( a \) What is the average number of costumers in the shop?

\( b \) What is the proportion of potential costumers that enter the shop?

\( c \) If the barber could work twice as fast, how much more business would he do (in average)?
Solution: (a) We model this situation with a continuous time Markov chain $X_t$, describing the number of customers in the shop at time $t$, which takes the possible values 0, 1, 2. If $X_t < 2$ the process can make a jump of +1 with rate 3/hour, while if $X_t > 0$ the process can make a jump of −1 with rate 4/hour (Recall: the mean value of an exponential random variable of rate $\lambda$ is $1/\lambda$). Hence, the infinitesimal generator matrix of this process is:

$$A = \begin{pmatrix} -3 & 3 & 0 \\ 4 & -7 & 3 \\ 0 & 4 & -4 \end{pmatrix}.$$ 

The corresponding equilibrium distribution $\bar{\pi}$ is solution of $\bar{\pi}A = 0$. Solving the system of equations we get:

$$\bar{\pi}_1 = \frac{3}{4} \bar{\pi}_0, \quad \bar{\pi}_2 = \frac{3}{4} \bar{\pi}_1,$$

and normalizing, we get

$$\bar{\pi}_0 = \frac{4^2}{4^3 - 3^3}, \quad \bar{\pi}_1 = \frac{3 \cdot 4}{4^3 - 3^3}, \quad \bar{\pi}_2 = \frac{3^2}{4^3 - 3^3}.$$

Hence, the average numbers of costumers in the shop (in the long run) is:

$$\bar{n} = 0\bar{\pi}_0 + 1\bar{\pi}_1 + 2\bar{\pi}_2 = \frac{3 \cdot 4}{4^3 - 3^3} + 2 \cdot \frac{3^2}{4^3 - 3^3} = \frac{30}{37}.$$

(b) The (average) proportion of potential costumers entering the shop is equal to the fraction of time that the shop has 0 or 1 costumers, which is the same as

$$\bar{\pi}_0 + \bar{\pi}_1 = \frac{4^2}{4^3 - 3^3} + \frac{3 \cdot 4}{4^3 - 3^3} = \frac{28}{37} \approx 0.76.$$

(c) If the barber works at double speed, the new equilibrium distribution is

$$\bar{\pi}_0' = 5 \cdot \frac{8^2}{8^3 - 3^3}, \quad \bar{\pi}_1' = 5 \cdot \frac{3 \cdot 8}{8^3 - 3^3}, \quad \bar{\pi}_2' = 5 \cdot \frac{3^2}{8^3 - 3^3}.$$

The (average) proportion of potential costumers entering the shop is equal to

$$\bar{\pi}_0' + \bar{\pi}_1' = 5 \cdot \frac{8^2}{8^3 - 3^3} + 5 \cdot \frac{3 \cdot 8}{8^3 - 3^3} = \frac{440}{485} \approx 0.91.$$

Hence, his business has a growth of 0.91/0.76 $\approx$ 1.19, approximately of %20.

Problem 3: Let $M_n, n \geq 0$, with $M_0 = 0$, be a martingale, and let $X_n = M_n - M_{n-1}, n \geq 1$. Prove that $\text{Var}(M_n) = \sum_{i=0}^{n} \text{Var}(X_i)$.

Solution: First, we note that $\mathbb{E}[M_n] = 0$, and $\mathbb{E}[X_n] = \mathbb{E}[M_n - M_{n-1}] = 0$, by definition of Martingale. Moreover, for $m < n$, we have

$$\mathbb{E}[X_nX_m] = \mathbb{E}[(M_m - M_{m-1})(M_n - M_{n-1})] = \mathbb{E}[\mathbb{E}[(M_m - M_{m-1})(M_n - M_{n-1})|\mathcal{F}_m]]$$

$$= \mathbb{E}[(M_m - M_{m-1})\mathbb{E}[(M_n - M_{n-1})|\mathcal{F}_m]] = \mathbb{E}[(M_m - M_{m-1})(M_n - M_m)] = 0.$$ 

We can write $M_n$ as a telescopic sum, to get

$$M_n = M_n - M_0 = (M_n - M_{n-1}) + (M_{n-1} - M_{n-2}) + \cdots + (M_1 - M_0) = X_1 + \cdots + X_n.$$ 

Hence,

$$\text{Var}(M_n) = \mathbb{E}[M_n^2] = \mathbb{E}[(\sum_{i=1}^{n} X_i)^2] = \sum_{i=1}^{n} \mathbb{E}X_i^2 + 2 \sum_{i<j} \mathbb{E}[X_iX_j] = \sum_{i=1}^{n} \text{Var}(X_i).$$

Problem 4: Let $X_1, X_2, X_3, \ldots$ be independent identically distributed random variables. Let $m(t) = \mathbb{E}[e^{tX_1}] < \infty$ be the moment generating function of $X_1$. Show that

$$M_n = m(t)^{-n}e^{t(X_1+\cdots+X_n)}, \quad n \geq 0,$$

is a martingale.

Solution: If $\mathcal{F}_n$ denotes the “information contained in the r.v.'s $X_1, \ldots, X_n$”, clearly $M_n$ is $\mathcal{F}_n$-measurable. We want to show that $\mathbb{E}[M_{n+1}|\mathcal{F}_n] = M_n$. We have

$$\mathbb{E}[M_{n+1}|\mathcal{F}_n] = \mathbb{E}[m(t)^{-n-1}e^{t(X_1+\cdots+X_n+X_{n+1})}|\mathcal{F}_n] = m(t)^{-n-1}\mathbb{E}[e^{t(X_1+\cdots+X_{n+1})}|\mathcal{F}_n]$$

$$= m(t)^{-n-1}e^{t(X_1+\cdots+X_{n+1})}\mathbb{E}[e^{tX_{n+1}}|\mathcal{F}_n] = m(t)^{-n-1}e^{t(X_1+\cdots+X_{n+1})}m(t) = m(t)^{-n}e^{t(X_1+\cdots+X_{n+1})} = M_n.$$
**Problem 5:** Let $B_t \geq 0$ be the standard Brownian motion. Find the probability density function of the following random variables:

1. $|B_t|$
2. $\max_{0 \leq s \leq t} B_s$
3. $\max_{0 \leq s \leq t} B_s - B_t$

**Solution:**

(a) $B_t$ is a normal random variable of variance $t$. Hence

$$P[|B_t| < x] = P[-x < B_t < x] = 2P[0 \leq B_t < x] = 2 \int_0^x \frac{1}{\sqrt{2\pi t}} e^{-y^2/2t} dy,$$

and the corresponding density function is

$$f_{|B_t|}(x) = \frac{d}{dx} P[|B_t| < x] = \frac{2}{\sqrt{2\pi t}} e^{-x^2/2t}.$$

(b) Since $B_0 = 0$, we clearly have $\max_{0 \leq s \leq t} B_s \geq 0$. By the reflection principle, for $x > 0$

$$P[\max_{0 \leq s \leq t} B_s \geq x] = 2P[B_t \geq x].$$

Hence,

$$P[\max_{0 \leq s \leq t} B_s < x] = 1 - P[\max_{0 \leq s \leq t} B_s \geq x] = 1 - 2P[B_t \geq x] = 1 - 2(1 - P[B_t < x]) = 2P[B_t < x] - 1,$$

and

$$f_{\max_{0 \leq s \leq t} B_s}(x) = \frac{d}{dx} P[\max_{0 \leq s \leq t} B_s < x] = \frac{d}{dx} (2P[B_t < x] - 1) = 2 \frac{d}{dx} \left( \int_0^x e^{-y^2/2t} dy \right) = \frac{2}{\sqrt{2\pi t}} e^{-x^2/2t}.$$

(c) We have, for $x \geq 0$,

$$P[\max_{0 \leq s \leq t} B_s \geq B_t + x] = \int_{x}^{\infty} P[\max_{0 \leq s \leq t} B_s \geq a + x | B_t = a] f_{B_t}(a) da = \int_{x}^{\infty} f_{B_t}(a) da.$$

With an argument similar to the one used to prove the reflection principle, it is not hard to see that

$$P[\max_{0 \leq s \leq t} B_s \geq a + x | B_t = a] = 1 - P[\max_{0 \leq s \leq t} B_s \leq a + x | B_t = a] = 1 - \int_{-\infty}^{a+x} f_{B_t}(a) da.$$

Hence,

$$P[\max_{0 \leq s \leq t} B_s \geq B_t + x] = \int_{-\infty}^{x} f_{B_t}(a) da + \int_{x}^{\infty} f_{B_t}(a + 2x) da = \int_{-\infty}^{x} f_{B_t}(a) da + \int_{x}^{\infty} f_{B_t}(a + 2x) da = 1 - \int_{-\infty}^{x} f_{B_t}(a) da.$$

Hence,

$$f_{\max_{0 \leq s \leq t} B_s - B_t}(x) = \frac{d}{dx} (1 - P[\max_{0 \leq s \leq t} B_s - B_t \geq x]) = \frac{2}{\sqrt{2\pi t}} e^{-x^2/2t}.$$

In conclusion, all three variables in (a), (b) and (c) have the same probability density.

**Problem 6:** Let $X_t = e^{at+bB_t}$, where $B_t$ be the standard Brownian motion, and $a, b \geq 0$ are constants.

1. Find the probability density function of $X_t$.
2. Compute $dX_t$.
3. For which values of $a, b$ is $X_t$ a martingale?

**Solution:**

(a) We have

$$P[X_t \leq x] = P[e^{at+bB_t} \leq x] = P[at + bB_t \leq \log x] = P[B_t \leq \frac{\log x - at}{b}] = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\frac{\log x - at}{b}} e^{-y^2/2t} dy.$$

Computing the derivative, we thus get

$$f_{X_t} = \frac{d}{dx} P[X_t \leq x] = \frac{1}{\sqrt{2\pi t}} e^{-\left(\frac{\log x - at}{b}\right)^2/2b^2 t} \frac{1}{bx}.$$
(b) By Itô formula, we have
\[ dX_t = \frac{\partial}{\partial B_t} X_t dB_t + \left( \frac{d}{dt} X_t + \frac{1}{2} \frac{\partial^2}{\partial B_t^2} X_t \right) dt = bX_t dB_t + \left( a + \frac{1}{2} b^2 \right) X_t dt. \]

(c) \( X_t \) is a Martingale when the \( dt \) term disappears, i.e. when \( a = -\frac{1}{2} b^2 \).