Your Name: Alberto de Sole

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Good job!
Problem 1:

Let \( X_t, t \geq 0 \), be a continuous time Markov chain on \( S = \{1, 2, 3\} \) with \( X_0 = 1 \) and infinitesimal generator matrix

\[
A = \begin{bmatrix}
-2 & 2 & 0 \\
1 & -3 & 2 \\
0 & 4 & -4
\end{bmatrix}.
\]

(a) Compute \( \frac{d}{dt} \mathbb{P}[X_t = n] \big|_{t=0} \) for \( n = 1, 2, 3 \).
(b) Compute \( \mathbb{P}[X_t = 3 \mid X_0 = 1] \) in the limit for \( t \to \infty \).
(c) Let \( \theta^{(3)} \) be the time of first passage in state 3: \( \theta^{(3)} = \inf \{ t \geq 0 \mid X_t = 3 \} \). Assuming \( X_0 = 1 \), compute \( \tau_1^{(3)} := \mathbb{E}[\theta^{(3)}] \), the mean passage time at 3.

Solution:

\[\begin{align*}
\text{(a) } & \quad \frac{d}{dt} \mathbb{P}[X_t = n] \big|_{t=0} = A \begin{pmatrix} 1 \n \n \end{pmatrix} = \begin{pmatrix} -2 & 2 & 0 \\
1 & -3 & 2 \\
0 & 4 & -4
\end{pmatrix} \begin{pmatrix} 1 \n \n \end{pmatrix} = 0 \\
\text{(b) } & \quad \mathbb{P}[X_t = 3 \mid X_0 = 1] \xrightarrow{t \to \infty} \begin{pmatrix} \frac{1}{4} \n \frac{1}{2} \n \frac{1}{4} \end{pmatrix} \\
\text{(c) } & \quad \tau_1^{(3)} = -A^{(-1)} \begin{pmatrix} 1 \n 1 \n 1 \end{pmatrix} = -\begin{pmatrix} -2 & 2 & 0 \\
1 & -3 & 2 \\
0 & 4 & -4
\end{pmatrix}^{-1} \begin{pmatrix} 1 \n 1 \n 1 \end{pmatrix} = -\frac{1}{4} \begin{pmatrix} -3 & -2 \\
-1 & -2 \\
0 & 4
\end{pmatrix} \begin{pmatrix} 1 \n 1 \n 1 \end{pmatrix} = \begin{pmatrix} 5/4 \n 3/4 \n 1/4 \end{pmatrix}
\end{align*}\]

Answer:
(a) \( \frac{d}{dt} \mathbb{P}[X_t = 1] \big|_{t=0} = -2 \), \( \frac{d}{dt} \mathbb{P}[X_t = 2] \big|_{t=0} = 2 \), \( \frac{d}{dt} \mathbb{P}[X_t = 3] \big|_{t=0} = 0 \)
(b) \( \lim_{t \to \infty} \mathbb{P}[X_t = 3 \mid X_0 = 1] = 1/4 \)
(c) \( \tau_1^{(3)} = 5/4 \)
Problem 2:
(a) Let $X_t$ be a birth and death process with rates

$$\lambda_n = 1 + \frac{1}{n+1} \quad \forall n \geq 0, \quad \mu_n = 1 \quad \forall n \geq 1.$$

Decide whether the process is transient, null recurrent (i.e. recurrent with no invariant distribution), or positive recurrent (i.e. recurrent with invariant distribution).

(b) Let $X_t$ be a birth and death process with rates

$$\lambda_n = 1 - \frac{2}{n+3} \quad \forall n \geq 0, \quad \mu_n = 1 \quad \forall n \geq 1.$$

Decide whether the process is transient, null recurrent, or positive recurrent.

Solution:

\[ \sum \frac{\mu_1 \cdots \mu_n}{\lambda_1 \cdots \lambda_n} < \infty \]

- **Recurrent** $\iff$ $\sum \frac{\mu_1 \cdots \mu_n}{\lambda_1 \cdots \lambda_n} < \infty$

- **Positive rec.** $\iff$ $\sum \frac{\mu_1 \cdots \mu_n}{\lambda_1 \cdots \lambda_n} < \infty$

\[ \sum \frac{1}{(1+\frac{1}{n+1}) \cdots (1+\frac{1}{n+1})} \]

\[ = \sum \frac{1}{(1+\frac{1}{n+1}) \cdots (1+\frac{1}{n+1})} = \sum \frac{2}{n+1} = +\infty \quad \Rightarrow \text{Recurrent} \]

\[ \sum \frac{\mu_1 \cdots \mu_{n-1}}{\lambda_1 \cdots \lambda_n} = \sum \frac{\mu_1 \cdots \mu_{n-1}}{\lambda_1 \cdots \lambda_n} = \sum \frac{2}{n} = +\infty \quad \Rightarrow \text{Null Rec.} \]

\[ \sum \frac{\mu_1 \cdots \mu_n}{\lambda_1 \cdots \lambda_n} = \sum \frac{1}{(1-\frac{2}{n+1}) \cdots (1-\frac{2}{n+1})} \cdots (1-\frac{2}{n+1}) (1-\frac{2}{n+3}) \]

\[ = \sum \frac{1}{(1-\frac{2}{n+1}) \cdots (1-\frac{2}{n+1})} = \sum \frac{2(n+2)(n+3)}{6} = +\infty \quad \Rightarrow \text{Rec.} \]

\[ \sum \frac{\mu_1 \cdots \mu_n}{\lambda_1 \cdots \lambda_n} = \sum \frac{\mu_1 \cdots \mu_n}{\lambda_1 \cdots \lambda_n} < \infty \quad \Rightarrow \text{Pos. Rec.} \]

Answer:
(a) $X_t$: transient \null recurrent \positive recurrent (circle one answer)

(b) $X_t$: transient \null recurrent \positive recurrent
Problem 3: 
In this exercise, $X_n$, $n = 0, 1, 2, \ldots$ denotes the symmetric simple ($p = 1/2$) random walk on $\mathbb{Z}$ with $X_0 = 0$. For which value(s) of the parameter $\alpha \geq 0$ each of the following stochastic processes is a martingale?

(i) $3X_n + \alpha$,
(ii) $X_n^2 - \alpha$,
(iii) $X_n^2 - \alpha n$,
(iv) $e^{X_n} - \alpha n$,
(v) $X_{\min(\tau_n, n)}$, where $\tau_n = \inf\{n \geq 0 \mid X_n = \alpha\}$ (here $\alpha$ is a positive integer).

Solution:

(i) $\mathbb{E}[M(\tau_n) \mid F_n] = \mathbb{E}[3X_n + \alpha \mid F_n] = \mathbb{E}[3X_n + \alpha \mid \mathbb{E}[3X_n^2 + \alpha^2 \mid F_n] = M_n$

(vi) $\mathbb{E}[X_n^2 - \alpha n \mid F_n] = \mathbb{E}[X_n^2 + \alpha n \mid F_n] - \alpha n$

= $X_n^2 + 2X_n\mathbb{E}[\varepsilon_n^2 \mid F_n] + \mathbb{E}[\varepsilon_n^2 \mid F_n] - \alpha n$

= $X_n^2 + 1 - \alpha_n = M_n + \alpha_n + 1 - \alpha_n

\Rightarrow \alpha_n = \alpha + 1$

Answer:

(i) $3X_n + \alpha$ is a martingale for $\alpha = \mathbb{R}

(ii) $X_n^2 - \alpha$ is a martingale for $\alpha = \mathbb{R}

(iii) $X_n^2 - \alpha n$ is a martingale for $\alpha = 0$

(iv) $e^{X_n} - \alpha n$ is a martingale for $\alpha = \frac{\phi \varepsilon + e^{-\phi}}{2}$

(v) $X_{\min(\tau_n, n)}$ is a martingale for $\alpha = \mathbb{R}$
Problem 4:

A betting game goes as follows. At each time $n$, I roll a fair dice (with values $1, 2, \ldots, 6$), if the result of the dice is less than 6 (in this case I set $X_n = -1$), I loose $\$1$; if the result of the dice is 6 (in this case I set $X_n = +1$), I win $\$x$.

(a) Let $M_n$ be the total winnings/losses at time $n$. Find a formula for $M_n$ in terms of the results $X_1, \ldots, X_n \in \{ \pm 1 \}$ up to time $n$.

(b) Find the value of $x$ for which $M_n$ is a martingale (i.e. the betting game is “fair”).

(c) Let now $x$ be the value found in part (b). Suppose that I stop playing either when $M_n \leq -10$ (i.e. I loose all the $\$10$ that I had when I started playing), or when $M_n = 100$ (i.e. I win at least $\$100$), and let $M$ be the total winnings/losses when I stop playing. What is $E[M]$?

Solution:

\[ x_1, x_2, x_3, \ldots \ i.i.d. \text{ Bernoulli } \epsilon \{ \pm 1 \} \text{ with} \]
\[ p = P[X_n = 1] = \frac{1}{6} \quad \text{and} \quad 1 - p = P[X_n = -1] = \frac{5}{6} \]

(A) \[ M_n = \sum_{i=0}^{n} X_i = \left\{ 1 \text{ if } X_i = -1 \right\} + X_i \quad \left\{ 1 \text{ if } X_i = +1 \right\} \]
\[ = \sum_{i=1}^{n} X_i \quad \text{where} \quad \sum_{i=1}^{n} X_i = \frac{X_{n+1}}{2} \sum_{i=1}^{n} X_i + \frac{5n}{n} \]

(B) \[ E[M_n | F_n] = E[ X_{n+1} \sum_{i=1}^{n} X_i + \frac{5n}{n} | F_n] = \]
\[ = \frac{X_{n+1}}{2} \sum_{i=1}^{n} X_i + \frac{5n}{n} \left( E[X_{n+1}] \right) \]
\[ = \frac{X_{n+1}}{2} \sum_{i=1}^{n} X_i + \frac{5n}{n} \quad \left( \frac{1}{6} \right) = 0 \]
\[ \Rightarrow \frac{X_{n+1}}{2} = -\frac{5n}{6} \quad \left( \Rightarrow \frac{X_{n+1}}{2} = \frac{5}{3} \right) \]

(c) We can apply the O.S.T.: \[ E[M] = E[M_{10}] = \sum_{i=1}^{10} E[M_i] = 0 \]

Answer:

(a) \[ \sum_{i=1}^{n} \left( \frac{X_{n+1}}{2} X_i + \frac{5n}{n} \right) = \frac{X_{n+1}}{2} \sum_{i=1}^{n} X_i + \frac{5n}{n} \cdot n \]

(b) \[ x = 5 \]

(c) \[ E[M] = 0 \]
Problem 5:

Consider the stochastic process \( X_t = 3 + 2B_t \), \( t \in [0, \infty) \), where \( B_t \) denotes the standard Brownian motion.

(a) Find the probability density function of the process \( X_t \), at time \( t \).

(b) Compute \( \mathbb{P}[X_t = 1] \) for some \( s \in [0, t] \).

(c) Let \( \tau_1 \) be the first time that \( X_t \leq 1 \). Find the probability density function of \( \tau_1 \).

Solution:

(a) \( \int_{-\infty}^{\frac{x^3}{2}} \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}} dy \)

(b) \( \frac{d}{dx} \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}} dy \)

(c) \( \int_{-\infty}^{\frac{x^3}{2}} \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}} dy \)

Answer:

(a) \( f_{X_t}(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-3)^2}{8t}} \)

(b) \( \mathbb{P}[X_s = 1] = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\frac{1}{2}} e^{-\frac{y^2}{2t}} dy \)

(c) \( f_{\tau_1}(t) = \frac{d}{dt} \int_{-\infty}^{\frac{1}{2} - \frac{1}{2t} x^2} \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}} dy \)