Problem Set Number 9, 18.385j/2.036j
MIT (Fall 2012)

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Due Wed., December 12, 2012. Turn it in at the Math. Students Office (Room 2-285) before 3 PM. There will be a box there for the problem set.

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1 Problem 09.02.02 - Strogatz.
   An ellipsoidal trapping region for the Lorenz attractor.

Statement for problem 09.02.02.
(An ellipsoidal trapping region for the Lorenz attractor.)

Consider the Lorenz system of equations:
\[
\frac{dx}{dt} = \sigma (y - x), \quad \frac{dy}{dt} = r x - y - x z, \quad \text{and} \quad \frac{dz}{dt} = x y - b z, \tag{1.1}
\]

where \( \sigma, r \) and \( b \) are positive constants. Let \( \Omega \) be the ellipsoidal region given by the points satisfying the equation
\[
E = r x^2 + \sigma y^2 + \sigma (z - 2 r)^2 \leq C, \tag{1.2}
\]

where \( C > 0 \) is a constant. Show that, for \( C \) large enough, all trajectories of the Lorenz equations eventually enter \( \Omega \) and stay there forever. Hint: compute \( dE/dt \).

For a much stiffer challenge: try to obtain the smallest possible value of \( C \) with this property.

Statement for problem 10.01.12.

(Newton’s Method).
Suppose that you want to find the roots of an equation \( g(x) = 0 \). Then Newton’s method says that you should consider the map \( x_{n+1} = f(x_n) \), where

\[
f(x) = x - \frac{g(x)}{g'(x)}. \tag{2.1}
\]

a. To calibrate the method, write down the Newton map \( x_{n+1} = f(x_n) \), for the equation \( g(x) = x^2 - 4 = 0 \).

b. Show that the Newton map has fixed points at \( x^* = \pm 2 \).

c. Show that these points are superstable.

d. Iterate the mapping numerically, starting from \( x_0 = 1 \). Note the extremely rapid convergence to the right answer!

Remark 2.0.1 The idea behind the method is this: Assume that you have an approximate solution \( x_a \) of \( g(x) = 0 \). To improve it write \( x_b = x_a + \delta x \), where \( \delta x \) is small. Then

\[
0 = g(x_b) = g(x_a + \delta x) \approx g(x_a) + g'(x_a) \delta x.
\]

Thus \( \delta x \approx -\frac{g(x_a)}{g'(x_a)} \), and the Newton map follows: \( x_b = f(x_a) \).

3 Problem 10.02.05 - Strogatz.

Orbit diagram for the exponential 1-D map.

Statement for problem 10.02.05.

(Orbit diagram for the exponential 1-D map).
Plot the orbit diagram for the following one dimensional map \( (\mathbb{R} \rightarrow \mathbb{R}) \):

\[
x_{n+1} = \exp(-rx_n), \tag{3.1}
\]

where \(-\infty < r < \infty\) is a parameter. Be sure to use a large enough range for both \( r \) and \( x \) to include the main features of interest. Also, try different initial conditions, just in case it matters.

Hint. In this case, you’ll find just one period doubling bifurcation and the show will be over. Justify analytically what you see.
4 Problem 11.03.08 - Strogatz. Sierpinski’s carpet.

Statement for problem 11.03.08.

(Sierpinski’s carpet). Consider the process shown in figure 4.1. The closed unit box is divided into nine equal boxes, and the open central box is deleted. Then this process is repeated for each of the eight remaining sub-boxes, and so on. Figure 4.1 shows the first two stages.

**A.** Sketch the next stage, $S_3$.

**B.** Find the similarity dimension of the limiting fractal, known as the Sierpinski carpet.

**C.** Show that the Sierpinski carpet has zero area.

![Sierpinski carpet construction](image)

Figure 4.1: Problem 11.3.8. Sierpinski carpet construction. The areas shaded in black are the parts of the original square deleted at each stage of the fractal's construction.

5 Bifurcations of a Critical Point for a 1-D map.

Statement: Bifurcations of a Critical Point for a 1D map.

When studying bifurcations of critical points in the lectures, we argued that we could understand most of the relevant possibilities simply by looking at 1-D flows. The argument was based on the idea that, generally, a stable critical point will become unstable in only one direction at a bifurcation — so that the flow will be trivial in all the other directions, and we need to concentrate only on what occurs in the unstable direction. The only important situation that is missed by this argument, is the case when two complex conjugate eigenvalues become unstable (Hopf bifurcation). Because in real valued
systems eigenvalues arise either in complex conjugate pairs or as single real ones, the cases where only one (or a complex pair) go unstable are the the only ones likely to occur (barring situations with special symmetries that “lock” eigenvalues into synchronous behavior).

The same argument can be made when studying bifurcations of limit cycles in any number of dimensions. In this case one considers the Poincaré map near the limit cycle,1 with the role of the eigenvalues taken over by the Floquet multipliers. Again, we can argue that we can understand a good deal of what happens by replacing the (multi-dimensional) Poincaré map by a one dimensional map with a stable fixed point, and then asking what can happen if the fixed point changes stability as some parameter is varied. In this problem you will be asked to do this.

Remark 5.0.1 Again, some important cases are missed by this approach. Namely: the case where a pair of complex Floquet multipliers becomes unstable (a Hopf bifurcation of a limit cycle) and the cases where a bifurcation occurs because of an interaction of the limit cycle with some other object (say, a critical point). Several examples of these situations can be found in Strogatz’ book, in section 8.4 (Global Bifurcations of Cycles).

Consider a one dimensional (smooth) map from the real line to itself $x \rightarrow y = f(x, \mu)$ (5.1) that depends on some (real valued) parameter $\mu$. Assume that $x = 0$ is a fixed point for all values of $\mu$ — that is, $f(0, \mu) \equiv 0$. Furthermore, assume that $x = 0$ is stable for $\mu < 0$ and unstable for $\mu > 0$. That is:

$$\left| \frac{\partial f}{\partial x}(0, \mu) \right| < 1 \quad \text{for} \quad \mu < 0, \quad \text{and}$$

$$\left| \frac{\partial f}{\partial x}(0, \mu) \right| > 1 \quad \text{for} \quad \mu > 0. \quad (5.3)$$

A further assumption, that involves no loss of generality (since the parameter $\mu$ can always be re-defined to make it true) is that

$$\frac{\partial^2 f}{\partial x \partial \mu}(0, 0) \neq 0. \quad (5.4)$$

This guarantees that the loss of stability is linear in $\mu$, as $\mu$ crosses zero. This is what is called a transversality condition. It means this:

Graph of the Floquet multiplier $\frac{\partial f}{\partial x}(0, \mu)$ as a function of $\mu$. Then the resulting curve crosses one of the lines $y = \pm 1$ transversally (curves not tangent at the common point) for $\mu = 0$.

By doing an appropriate expansion of the map $f$ for $x$ and $\mu$ small (or by any other means), show that (generally2) the following happens:

a. For $\frac{\partial f}{\partial x}(0, 0) = 1$, either:

a1. Transcritical bifurcation (no special symmetries assumed for $f$): There exists another fixed point, $x_* = x_*(\mu) = O(\mu)$, such that: $x_* \neq 0$ is unstable for $\mu < 0$ and $x_* \neq 0$ is stable for $\mu > 0$. The two points “collide” at $\mu = 0$ and exchange stability.

1 The limit cycle is a fixed point for this map.

2 There are special conditions under which all this fails. You must find them as part of your analysis. What are they?
a2. Supercritical or soft pitchfork bifurcation, assuming that \( f \) is an odd function of \( x \): Two stable fixed points exist for \( \mu > 0 \), one on each side of \( x = 0 \), at a distance \( O(\sqrt{\mu}) \). All three points merge for \( \mu = 0 \).

a3. Subcritical or hard pitchfork bifurcation, assuming that \( f \) is an odd function of \( x \): Two unstable fixed points exist for \( \mu < 0 \), one on each side of \( x = 0 \), at a distance \( O(\sqrt{-\mu}) \). All three points merge for \( \mu = 0 \).

What does all this mean in the context of the Poincaré map for a limit cycle?

b. For \( \frac{\partial f}{\partial x}(0, 0) = -1 \) (no special symmetries assumed for \( f \)), either:

b1. Supercritical or soft flip bifurcation: For \( \mu > 0 \) two points \( x_1(\mu) \approx -x_2(\mu) = O(\sqrt{\mu}) \) exist, on each side of the fixed point \( x = 0 \), with \( x_2 = f(x_1, \mu) \) and \( x_1 = f(x_2, \mu) \). Thus \( \{x_1, x_2\} \) is a period two orbit for the map (5.1). Show that this orbit is stable.

b2. Subcritical or hard flip bifurcation: For \( \mu < 0 \) two points \( x_1(\mu) \approx -x_2(\mu) = O(\sqrt{-\mu}) \) exist, on each side of the fixed point \( x = 0 \), with \( x_2 = f(x_1, \mu) \) and \( x_1 = f(x_2, \mu) \). Thus \( \{x_1, x_2\} \) is a period two orbit for the map (5.1). Show that this orbit is unstable.

In the context of the Poincaré map for a limit cycle, a flip bifurcation corresponds to a period doubling bifurcation of the limit cycle.

Hint 5.0.1 If you expand \( f \) in a Taylor expansion near \( x = 0 \) and \( \mu = 0 \), up to the leading order beyond the trivial first term (\( f \sim \pm x \)), and you make sure to keep all the relevant terms (and nothing else), and you make sure to identify certain terms that must vanish, the problem reduces to (mostly) trivial high-school algebra. In figuring out what to keep and what not to keep, note that you have two small quantities (\( x \) and \( \mu \)), whose sizes are related. The process is very similar to the Hopf bifurcation expansion calculation (e.g. see course notes), but much simpler computationally.

For part (b) you will have to keep in the expansion not just the leading order terms beyond the trivial first term, but the next order as well. Reason: In this case, you will be looking for solutions to the equation \( f(f(x, \mu), \mu) = x \). But, when you calculate \( f(f(x, \mu), \mu) \), you will see that the second order terms cancel out — thus the need for an extra term in the expansion. This is the same phenomena that forces the Hopf bifurcation calculation to third order.

\[ \text{\textbullet\quad VERY IMPORTANT.} \]

To standardize the notation, use the following symbols in your answer:

\[ \nu = \frac{\partial f}{\partial x}(0, 0) \quad \text{— note that } \nu = \pm 1, \]

\[ a = \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(0, 0), \quad b = \frac{\partial^2 f}{\partial x \partial \mu}(0, 0), \quad c = \frac{1}{6} \frac{\partial^3 f}{\partial x^3}(0, 0), \quad \text{and} \quad d = \frac{1}{2} \frac{\partial^3 f}{\partial x^2 \partial \mu}(0, 0). \]

Then obtain leading order expressions for the fixed points and flip-bifurcation orbits in terms of these quantities.

THE END.