Problem Set Number 8, 18.385j/2.036j
MIT (Fall 2012)

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November 11, 2012

Due Wed., November 28, 2012. Turn it in at the Math. Students Office (Room 2-285) before 3 PM. There will be a box there for the problem set.

Note: this is an UPDATE [2012-11-20] to correct some (small) typos.

Contents

1 Problem 07.06.18 - Strogatz (Mathieu eqn. and a super-slow time scale). 1
2 Notes: coupled oscillators, phase locking, etc. 2
   2.1 On phases and frequencies. ................................................................. 2
   2.2 Phase locking and oscillator death. ....................................................... 2
3 First order equation with a periodic right hand side. 3
4 Friends on a circular track run in opposite directions. 4
5 Hill equation problem # 01. 5

1 Problem 07.06.18 - Strogatz
(Mathieu equation and a super-slow time scale).

Statement for problem 07.06.18.
(Mathieu equation and a super-slow time scale). Consider the Mathieu equation

\[ \frac{d^2x}{dt^2} + (a + \epsilon \cos(t)) x = 0, \]  

(1.1)

with \( a \approx 1 \). Using two-timing with a slow time scale \( T = \epsilon^2 t \), show that the solution becomes unbounded as \( t \to \infty \), if

\[ 1 - \frac{1}{12} \epsilon^2 + O(\epsilon^4) \leq a \leq 1 + \frac{5}{12} \epsilon^2 + O(\epsilon^4). \]

Question: Why is \( a \approx 1 \) an interesting regime?

Hint. Assume that \( a = 1 + \epsilon^2 a_2 + \ldots \), expand, and look at the behavior of the solution as a function of the coefficient \( a_2 \).
2. Notes: coupled oscillators, phase locking, etc.

2.1 On phases and frequencies.

Consider a system made by two coupled oscillators, where each of the oscillators (when not coupled) has a stable attracting limit cycle. Let the limit cycle solutions for the two oscillators be given by \( \vec{x}_1 = \vec{F}_1(\omega_1 t) \) and \( \vec{x}_2 = \vec{F}_2(\omega_2 t) \), where \( \vec{x}_1 \) and \( \vec{x}_2 \) are the vectors of variables for each of the two systems, the \( \vec{F}_j \) are periodic functions of period \( 2\pi \), and the \( \omega_j \) are constants (related to the limit cycle periods by \( \omega_j = 2\pi/T_j \)). In the un-coupled system, the two limit cycle orbits make up a stable attracting invariant torus for the evolution. Assume now that either the coupling is weak, or that the two limit cycles are strongly stable. Then the stable attracting invariant torus survives for the coupled system.\(^1\)

The solutions (on this torus) can be (approximately) represented by

\[
\vec{x}_1 \approx \vec{F}_1(\theta_1) \quad \text{and} \quad \vec{x}_2 \approx \vec{F}_2(\theta_2).
\]

Here \( \theta_1 = \theta_1(t) \) and \( \theta_2 = \theta_2(t) \) satisfy some equations, of the general form

\[
\dot{\theta}_1 = \omega_1 + K_1(\theta_1, \theta_2) \quad \text{and} \quad \dot{\theta}_2 = \omega_2 + K_2(\theta_1, \theta_2),
\]

where \( K_1 \) and \( K_2 \) are the “projections” of the coupling terms along the oscillator limit cycles. For example, take \( K_1(\theta_1, \theta_2) = \sin \theta_1 \cos \theta_2 \) and \( K_2(\theta_1, \theta_2) = \sin \theta_2 \cos \theta_1 \). A second example can be found in § 8.6 of the book by Strogatz (Nonlinear Dynamics and Chaos), where a model system with \( K_1(\theta_1, \theta_2) = -\kappa_1 \sin(\theta_1 - \theta_2) \) and \( K_2(\theta_1, \theta_2) = \kappa_2 \sin(\theta_1 - \theta_2) \) is introduced (where \( \kappa_1, \kappa_2 > 0 \) are constants). Note that:

1. In (2.2), \( K_1 \) and \( K_2 \) are \( 2\pi \)-periodic functions of \( \theta_1 \) and \( \theta_2 \).

2. The phase space for (2.2) is the invariant torus \( T \), on which \( \theta_1 \) and \( \theta_2 \) are the angles. We can also think of \( T \) as a \( 2\pi \times 2\pi \) square with its opposite sides identified. On \( T \) a solution is periodic if and only if \( \theta_1(t + T) = \theta_1(t) + 2n\pi \) and \( \theta_2(t + T) = \theta_2(t) + 2m\pi \), where \( T > 0 \) is the period, and both \( n \) and \( m \) are integers.

3. In the “Coupled oscillators \# 01” problem an example of the process leading to (2.2) is presented.

4. The \( \theta_j \)’s are the oscillator phases. One can also define oscillator frequencies, even when the \( \theta_j \)’s do not have the form \( \theta_j = \omega_j t \), with \( \omega_j \) constant. The idea is that, near any time \( t_0 \) we can write \( \theta_j = \theta_j(t_0) + \dot{\theta}_j(t_0)(t - t_0) + \ldots \), identifying \( \dot{\theta}_j(t_0) \) as the local frequency. Hence, we define the oscillator frequencies by \( \dot{\omega}_j = \dot{\theta}_j \). These frequencies are, of course, generally not constants.

5. The notion of phases can survive even if the limit cycles cease to exist (i.e.: oscillator death). For example: if the equations for \( \theta_1 \) and \( \theta_2 \) have an attracting critical point. We will see examples where this happens in the problems, e.g.: “Bifurcations in the torus \# 01”.

2.2 Phase locking and oscillator death.

The coupling of two oscillators, each with a stable attracting limit cycle, can produce many behaviors. Two of particular interest are

\(^1\)With a (slightly) changed shape and position.
1. Often, if the frequencies are close enough, the system **phase locks**. This means that a stable periodic solution arises, in which both oscillators run at some composite frequency, with their phase difference kept constant. The composite frequency need not be constant. In fact, it may periodically oscillate about a constant average value.

2. However, the coupling may also suppress the oscillations, with the resulting system having a stable steady state. This even if none of the component oscillators has a stable steady state. This is **oscillator death**. It can happen not only for coupled pairs of oscillators, but also for chains of oscillators with coupling to the nearest neighbors.

On the other hand, we note that it is also possible to produce an oscillating system, with a stable oscillation, by coupling non-oscillating systems (e.g., coupling of excitable systems can do this).

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### 3 First order equation with a periodic right hand side.

First order equation with a periodic right hand side. **Statement.**

Consider the equation

\[
\dot{\phi} = 1 - \kappa \sin \phi, \quad \text{where} \quad 0 < \kappa < 1. \tag{3.1}
\]

Since \( \dot{\phi} \geq 1 - \kappa > 0 \), \( \phi \) is monotone increasing. **Prove the statements below.**

1. There is a constant \( 0 < \mu < 1 \), and a function \( \Phi = \Phi(\zeta) \) — periodic of period \( 2\pi \) — such that any solution to (3.1) has the form

\[
\phi = \mu (t - t_0) + \Phi(\mu (t - t_0)), \tag{3.2}
\]

where \( t_0 \) is a constant and \( \Phi(0) = 0 \).

2. Note that \( \sin(\phi) \) is periodic in \( t \), of period \( T = \frac{2\pi}{\mu} \), with

\[
M = \text{average}(\sin \phi) = \frac{1-\mu}{\kappa} > 0, \tag{3.3}
\]

where \( M \) is defined by the first equality — \( M = M(\kappa) \) only, since \( \mu \) depends on \( \kappa \) only.

3. Let \( \phi_* \) be the solution to (3.1) defined by \( \phi_*(0) = 0 \) — i.e.: set \( t_0 = 0 \) in (3.2). Then

\[
\Theta(\mu t) = \int_0^t (\sin(\phi_*(s)) - M) \, ds = \frac{1}{\kappa} \Phi(\mu t), \tag{3.4}
\]

where \( \Theta \) is defined by the first equality.

4. Assume that \( 0 < \kappa \ll 1 \). Then a Poincaré-Lindstedt expansion yields

\[
\phi_* = \mu t - \kappa (1 - \cos(\mu t)) + O(\kappa^2) \quad \text{and} \quad \mu = 1 - \frac{1}{2} \kappa^2 + O(\kappa^4). \tag{3.5}
\]

It follows that \( T = 2\pi + \pi \kappa^2 + O(\kappa^4) \) and \( M = \frac{1}{2} \kappa + O(\kappa^3) \).

5. Assume that \( 0 < 1 - \kappa \ll 1 \). Then

\[
\mu = O(\sqrt{1 - \kappa}) \tag{3.6}
\]
Hints.

a. Define $T > 0$ as the (unique) time at which $\phi_*(T) = 2\pi$ — why is the solution unique?

b. Show that $\phi_*(t + T) = 2\pi + \phi_*(t)$ — sub-hint: both sides are solutions!

c. Define $\Phi$ by $\Phi(\mu t) = \phi_*(t) - \mu t$, with $\mu = 2\pi/T$, and show that $\Phi$ is periodic of period $2\pi$.

d. Write the general solution in terms of $\phi_*$.

e. Show that $T = O\left(1/\sqrt{1 - \kappa}\right)$ as $\kappa \to 1$ — sub-hint: critical slowing-down.

f. To show that $\mu < 1$, use (3.1) and separation of variables to write $T$ as an integral over $\phi$ from 0 to $2\pi$. Then show $T > 2\pi$

g. To show (3.3), take the average of (3.1).

h. To obtain the second equality in (3.4), substitute $\phi_* = \mu t + \Phi(\mu t)$ into (3.1), and obtain a formula for $\sin(\phi_*)$ in terms of $\Phi$.

4 Friends on a circular track run in opposite directions.

Friends on a circular track run in opposite directions. Statement.

In § 8.6 of his book (Nonlinear Dynamics and Chaos) Strogatz introduces the following model equations in the torus

$$
\dot{\theta}_1 = \omega_1 - \kappa_1 \sin(\theta_1 - \theta_2) \quad \text{and} \quad \dot{\theta}_2 = \omega_2 + \kappa_2 \sin(\theta_1 - \theta_2),
$$

where: (a) $\theta_1, \theta_2$ are the phases of two oscillators, (b) $\omega_1, \omega_2 > 0$ are their natural frequencies, and (c) $\kappa_1, \kappa_2 > 0$ are the coupling constants. Then he gives a simple interpretation\(^2\) for the equations as modeling two friends jogging in a circular track of radius $R$, where $(\theta_1, \theta_2)$ are their (angular) positions on the track, $(R \omega_1, R \omega_2)$ are their preferred running speeds, and the coupling models their desire to run together. In this interpretation (4.1) corresponds to the two friends running along the track in the same direction (counter-clockwise). Imagine now a situation where they are running in opposite directions along the track.\(^3\) Then the equations have to be modified to

$$
\dot{\theta}_1 = \omega_1 - \kappa_1 \sin(\theta_1 - \theta_2) \quad \text{and} \quad \dot{\theta}_2 = -\omega_2 + \kappa_2 \sin(\theta_1 - \theta_2).
$$

Equivalently, with $\varphi_1 = \theta_1$ and $\varphi_2 = -\theta_2$,

$$
\dot{\varphi}_1 = \omega_1 - \kappa_1 \sin(\varphi_1 + \varphi_2) \quad \text{and} \quad \dot{\varphi}_2 = \omega_2 - \kappa_2 \sin(\varphi_1 + \varphi_2).
$$

The motivation given above for (4.2) is not-too-serious, but these equations — in the form given in (4.3) — are a particular example of the type of equations that result when very stable limit cycle oscillators are coupled — e.g., see the problem Coupled oscillators # 01.

Study the behavior of the system in (4.2). In particular: Find all the bifurcations. Is there phase locking? Note: since (4.2) is a system in the torus, bifurcations that do not appear in the plane might

\(^2\)These equations appear in other contexts, as mentioned by Strogatz. Also see § 2.

\(^3\)Coach’s orders, because otherwise they run together, talk, and do not train properly.
occur, look carefully. **Notation and units:** To simplify the answer, assume that the time unit is such that $\omega_1 + \omega_2 = 1$. In addition, use the notation $\kappa = \kappa_1 + \kappa_2 > 0$.

**Hint.** The results in “First order equation with a periodic right hand side” (§ 3) will be useful. Do not try to pin-point the various regimes and bifurcations using explicit formulas involving $\omega_1, \omega_2, \kappa_1$ and $\kappa_2$ only. You will also need either $M = M(\kappa)$ or $\mu = \mu(\kappa)$, defined in § 3.

## 5 Hill equation problem # 01.

### Hill equation problem # 01. Statement.

Define the periodic (of period $2\pi$) square wave $S = S(\xi)$ as follows

$$S(\xi + 2n\pi) = 1 \text{ for } 0 < \xi < \pi \quad \text{and} \quad S(\xi + 2n\pi) = 0 \text{ for } \pi < \xi < 2\pi,$$

where $n$ is an arbitrary integer. Consider now the Hill equation problem

$$\ddot{x} + a^2 S(\omega t) x = 0,$$

where $a > 0$ and $\omega > 0$ are constants — note that the **period is** $T = \frac{2\pi}{\omega}$. **Problem tasks:**

1. **Write the equations in the standard form** $\dot{X} = A(\omega t) X$, where $A$ is a $2 \times 2$ matrix with period $2\pi$ and $X$ is a two-vector.
2. **Find the fundamental solution** $U$ to the system derived in item 1 — the matrix solution with $U(0) = \text{identity}$.
3. **Compute the Floquet matrix** $R = U(T)$.
4. **Write the Floquet multipliers in terms of** $\alpha = \frac{1}{2} \text{Tr}(R)$. You should find that $\alpha$ is a function of a single parameter $\tau > 0$. That is: $\alpha = \alpha(\tau)$. **What is $\tau$?**
5. **Plot $\alpha$ versus $\tau$.**
6. **As $\tau$ grows from zero, there is a first threshold for instability:** $\tau = \tau_1$ — this means that the solutions to (5.2) are bounded for $0 < \tau < \tau_1$, but some solutions grow exponentially for $\tau$ slightly above $\tau_1$. **Compute $\tau_1$ (at least two significant digits).**
7. **For $\tau$ slightly above $\tau_1$, there is a solution to (5.2) that grows (slowly if $\tau$ is close to $\tau_1$). What is the frequency and period of this solution when $0 < \tau - \tau_1 \ll 1$?**

**Hint:** The Floquet solutions have the general form $x = \chi(\omega t) e^{\mu t}$, where $\chi$ is periodic of period $2\pi$, and $\mu$ is the Floquet exponent. If $\mu$ is pure imaginary, and rationally related to $\omega$, this solution is periodic — with some period $T_s$ and frequency $2\pi/T_s$. If $\mu$ has a small positive real part, and its imaginary part limits to a rational fraction of $\omega$, then the situation described in item 7 arises.

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**THE END.**

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4In case you wonder: There is no problem with the discontinuity in $S$ as far as existence and uniqueness because $a^2 S x$ is Lipschitz continuous with respect to $x$ — which is what matters.