1 Can this equation model a hanging chain? v01.

Statement: Can this equation model a hanging chain? v01.

Consider a rope (or a chain) hanging (at equilibrium) from two nails at the same height, the nails separated by some horizontal distance. Idealize the rope as a curve in space, and describe its shape by the vertical deviation of the rope \( u = u(x) \) from the horizontal straight line connecting the nails. Assume now that someone tells you that he has constructed a special rope, where the density and elastic properties of the rope along its length have been carefully arranged, so that the shape \( u = u(x) \) can be obtained by solving the following problem\(^1\)

\[
\frac{d^2 u}{dx^2} + a^2 u = 0, \quad \text{for } 0 < x < \pi, \quad \text{with } u(0) = u(\pi) = 0, \quad (1.0.1)
\]

\(^1\) Assume a-dimensional variables.
where \( a > 0 \) is some constant — resulting from the force of gravity and the rope properties: length, density, elastic strength, etc.

**Is this possible?** Explain and justify your answer.

Notes: (i) The rope is on Earth, there is a non-vanishing gravity force. (ii) The rope is made of real materials: infinite strength, rigid, etc., are not allowed. Why does this matter?

Hint #1: Note that the statement is not that (1.0.1) is “the physical model” for the rope shape. It only says that “the rope shape can be obtained by solving (1.0.1)”. Hence deriving all the possible physical models, and then checking to see if (1.0.1) is one of them is not enough. You would have to check if there is a model all whose solutions are equal to those of (1.0.1). This is far from the simplest way to answer this problem.

Hint #2: Solve (1.0.1) and see how it depends on the parameter \( a \).

### 2 Linear model for a hanging rope under tension.

**Statement:** Linear model for a hanging rope under tension.

Consider a thin rope stretched (at equilibrium) between two nails at the same height, with the nails separated by some horizontal distance. Idealize the rope as a curve in space, and describe its shape by the vertical deviation of the rope \( u = u(x) \) from the horizontal straight line connecting the nails. Assume that the deviations of the rope shape from a straight line are small, so that at any point along the rope you can do the approximation \( \sin(\theta) \approx \tan(\theta) = \frac{du}{dx} \), where \( \theta \) is the angle with the horizontal made by the tangent line to the rope.

Let the distance between the nails be \( L \) (the left nail is at \( x = 0 \) and the right one at \( x = L \)), let \( T > 0 \) be the tension along the rope,\(^2\) and let \( g \) be the acceleration of gravity.\(^3\) Finally assume that the rope is homogeneous, with density \( \rho \) (mass per unit length).

**Write a mathematical model for \( u \), and show that it is well posed.**

Hint #1: The model will be a boundary value problem, in \( 0 < x < L \), for a second order, non-homogeneous, linear ode for \( u \).

Hint #2: To derive the ode for \( u \), consider the balance of the vertical forces\(^4\) on an arbitrary segment of the rope \( a \leq x \leq b \). The forces are: (i) gravity, (ii) the tension force by the piece of rope on \( x > b \), and (iii) the tension force by the piece of rope on \( x > a \). Then divide by \((b - a)\) the resulting equation, and take the limit \((b - a) \to 0 \) — with both \( b, a \to x_0 = \) some arbitrary point along the rope.

---

\(^2\) The tension is the magnitude of the force with which, at any point, one side of the rope pulls on the other. The force vector is directed along the rope’s tangent, towards the pulling side, and has length \( T \).

\(^3\) Hence, on any mass \( m \), there is a downwards force of magnitude \( mg \).

\(^4\) Because the rope is at equilibrium, the forces must add to zero.
3 Ill posed initial value problem #01.

Statement: Ill posed initial value problem #01.

Consider the initial value problem
\[ \frac{dx}{dt} = 2 \sqrt{|x|}, \quad \text{for } 0 < t, \quad \text{with } x(0) = 0. \] (3.0.1)

Show that this problem has infinitely many solutions, that can be parameterized by the time \( 0 \leq \tau \leq \infty \) beyond which the solution ceases to vanish.

Hint: use separation of variables to find all the solutions that are possible when \( x \neq 0 \).

4 Implicit function problem #01.

Statement: Implicit function problem #01.

Consider the following equation
\[ f(x, \lambda) = x + \lambda \cos(x) = 0, \] (4.0.1)

with the particular solution \((x, \lambda) = (0, 0)\). Since \( f_x(0, 0) = 1 \), the implicit function theorem guarantees that: there is a neighborhood of \( \lambda = 0 \) where (4.0.1) has a unique solution, \( x = X(\lambda) \), such that \( X(0) = 0 \). Furthermore, since \( f \) is an analytic function, \( X \) is an analytic function of \( \lambda \). This means that \( X \) has a Taylor series
\[ X = \sum_{n=0}^{\infty} x_n \lambda^n, \] (4.0.2)

which converges for \( |\lambda| \) small enough. Find \( x_1, x_3, x_5, \) and \( x_n \) for all even \( n \).

5 Problem 02.02.08 - Strogatz
(Backwards: flow to equation).

Statement for problem 02.02.08.

(Working backwards, from flows to equations). Given an equation \( \dot{x} = f(x) \), we know how to sketch the corresponding flow on the real line. Here you are asked to solve the opposite problem:
6 Problem 02.02.09 - Strogatz
(Backwards: solutions to equation).

Statement for problem 02.02.09.

(Working backwards, from solutions to equations). Find an equation $\dot{x} = f(x)$ whose solutions $x = x(t)$ are consistent with those shown in Figure 0.1.

Note: This is (almost) the same as (02.02.8). The figure shows you that there are two critical points on $x \geq -1$, namely: $x = 0$ (stable) and $x = 1$ (unstable). Thus you must produce $\dot{x} = f(x)$ where: $f$ has zeros at $x = 0$ and $x = 1$; $f(x) < 0$ for $0 < x < 1$; and $f$ is positive elsewhere. Actually, what $f$ might do for $x < -1$ or $x > 1.7$ is not specified by the figure.
7 Get equation from phase line portrait problem #01.

Statement: Get equation from phase line portrait problem #01.

Consider the ode on the line
\[ \frac{dx}{dt} = f(x), \] (7.0.1)
where \( f \) is some function which is (at least) Lipschitz continuous. Assume that (7.0.1) has exactly two critical points (i.e.: \( x_1 \) and \( x_2 \), with \(-\infty < x_1 < x_2 < \infty\)). Assume also that both critical points are semi-stable.\(^5\) Is this possible? Does a function \( f = f(x) \) yielding this exist?
If the answer is no, prove it.
If the answer is yes, prove it by giving an example.

8 Get equation from phase line portrait problem #05.

Statement: Get equation from phase line portrait problem #05.

Consider the ode on the line
\[ \frac{dx}{dt} = f(x), \] (8.0.1)
where \( f \) is some function which is (at least) Lipschitz continuous. Assume that (8.0.1) has exactly two critical points (i.e.: \( x_1 \) and \( x_2 \), with \(-\infty < x_1 < x_2 < \infty\)). Assume also that both critical points are stable. Is this possible? Does a function \( f = f(x) \) yielding this exist?
If the answer is no, prove it.
If the answer is yes, prove it by giving an example.

9 Problem 02.06.01 - Strogatz (1D oscillator “paradox”).

Statement for problem 02.06.01.

Explain this paradox: a simple harmonic oscillator \( m\ddot{x} = -kx \) is a system that oscillates in one dimension (along the x-axis). But the text says that one-dimensional systems cannot oscillate.

THE END.

---

\(^5\) A critical point is semi-stable if the solutions diverge from the critical point on one side, and converge on the other.