Problem Set Number 5, 18.385j/2.036j
MIT (Fall 2010)

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Turn it in at the Math. Students Office (Room 2-108) before 2 PM.
There will be a box there for the problem set.

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1 Problem 07.05.05 - Strogatz
(Find relaxation oscillation period).

Statement for problem 07.05.05.

Consider the equation
\[ \frac{d^2 x}{dt^2} + \mu (|x| - 1) \frac{dx}{dt} + x = 0. \]  \hfill (1.1)

Find the approximate period of the limit cycle for \( \mu \gg 1 \).

Hint. This problem can be done using exactly the same set of tricks used to calculate the period for the van de Pol equation limit cycle in the relaxation oscillation regime.

Part 2. Assume now that \( 0 < \mu \ll 1 \). Find an approximation for the limit cycle and the period which is accurate up to \( O(\mu^2) \) error terms. In other words, propose a Poincaré–Lindstedt expansion of the form:

\[ x = x_0(T) + \mu x_1(T) + \mu^2 x_2(T) + \ldots, \]

where \( T = \omega t, \omega = 1 + \mu \omega_1 + \mu^2 \omega_2 + \ldots \), and the \( x_n \)'s are 2 \( \pi \)-periodic functions of \( T \). Then find \( \omega_1 \) and \( x_1 \).
Remark 1.1 This problem gives a (relatively) simple illustration of how to do a Poincaré–Lindstedt expansion in situations involving non-polynomial functions, so that Fourier Series techniques must be used to compute the resonances.

2 Problem 07.06.18 - Strogatz
(Mathieu equation and a super-slow time scale).

Statement for problem 07.06.18.
(Mathieu equation and a super-slow time scale). Consider the Mathieu equation
\[
\frac{d^2x}{dt^2} + (a + \epsilon \cos(t)) x = 0, \tag{2.1}
\]
with \( a \approx 1 \). Using two-timing with a slow time scale \( T = \epsilon^2 t \), show that the solution becomes unbounded as \( t \to \infty \), if
\[
1 - \frac{1}{12}\epsilon^2 + O(\epsilon^4) \leq a \leq 1 + \frac{5}{12}\epsilon^2 + O(\epsilon^4).
\]

Hint. Assume that \( a = 1 + \epsilon^2 a_2 + \ldots \), expand, and look at the behavior of the solution as a function of the coefficient \( a_2 \). Why is \( a \approx 1 \) an interesting regime?

3 Problem 08.01.06 - Strogatz
(Bifurcations in the phase plane).

Statement for problem 08.01.06.
(Bifurcations in the phase plane). Consider the system
\[
\frac{dx}{dt} = y - 2x, \quad \text{and} \quad \frac{dy}{dt} = \mu + x^2 - y, \tag{3.1}
\]
where \( \mu \) is a constant.

a) Sketch the nullclines.

b) Find and classify (ALL) the bifurcations that occur as \( \mu \) varies.

c) Sketch the phase portrait as a function of \( \mu \) — three plots per bifurcation: before, at, and after the bifurcation. Are there any limit cycles or cycle graphs?
4 Problem 08.02.07 - Strogatz  
(Hopf and homoclinic bifurcations using a computer).

Statement for problem 08.02.07.

(Hopf and homoclinic bifurcations using a computer). For the following system
\[
\frac{dx}{dt} = \mu x + y - x^2 \quad \text{and} \quad \frac{dy}{dt} = -x + \mu y + 2x^2, \tag{4.1}
\]
a Hopf bifurcation occurs at the origin when \( \mu = 0 \). Using a computer, plot the phase portrait and determine whether the bifurcation is subcritical or supercritical. For small values of \( \mu \), verify that the limit cycle is nearly circular. Then measure the period and radius of the limit cycle, and show that the radius \( R \) scales with \( \mu \) as predicted by theory.

In addition to a Hopf bifurcation, this system also exhibits an homoclinic bifurcation of the limit cycle. FIND IT.

5 Problem 08.02.12 - Strogatz  
(Analytical criterion for Hopf bifurcations).

Statement for problem 08.02.12.

(Analytical criterion to decide if a Hopf bifurcation is subcritical or supercritical). Any system at a Hopf bifurcation can be put in the following form by a suitable change of variables
\[
\frac{dx}{dt} = -\omega y + f(x, y) \quad \text{and} \quad \frac{dy}{dt} = \omega x + g(x, y), \quad \text{with} \quad \omega \neq 0, \tag{5.1}
\]
where \( f \) and \( g \) contain only higher-order nonlinear terms that vanish at the origin. As shown by Guckenheimer and Holmes\(^1\) (pp. 152-156), one can decide whether the bifurcation is subcritical or supercritical by calculating the sign of the following quantity:
\[
a = \frac{1}{16} \left\{ f_{xx} + f_{yy} + g_{xx} + g_{yy} + \frac{1}{\omega} \left[ f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx} g_{xx} + f_{yy} g_{yy} \right] \right\}, \tag{5.2}
\]
where the subscripts indicate partial derivatives evaluated at \((0, 0)\). The criterion is: If \( a < 0 \), the bifurcation is supercritical; if \( a > 0 \), the bifurcation is subcritical.

a. Calculate \( a \) for the system \( \dot{x} = -y + xy^2 \) and \( \dot{y} = x - x^2 \).

b. Use part a to decide what type of Hopf bifurcation occurs for \( \dot{x} = -y + \mu x + xy^2 \) and \( \dot{y} = x + \mu y - x^2 \) at \( \mu = 0 \). Then compare with the results of Exercises 8.2.2 and 8.2.4.

What does $a$ measure? Roughly speaking, $a$ is the coefficient of the cubic term in the equation $\dot{r} = a r^3$ governing the radial dynamics of the bifurcation. Here $r$ is a slightly transformed version of the usual polar coordinate. For details, see Guckenheimer and Holmes or Grimshaw.\(^2\)

**c. EXTRA QUESTION:** The criterion spelled above can be justified using a weakly nonlinear (two-times) asymptotic expansion for the solutions of equation (5.1), in the regime where the solutions have amplitude $0 < \epsilon \ll 1$. However, as one could easily guess from the complicated expression in (5.2), any justification of this criterion involves quite a lot of algebra. Fortunately, a much simpler calculation occurs when the quadratic nonlinearities vanish, while keeping the essence of the technique intact. Hence: justify the criteria above in the special case when the quadratic nonlinearities vanish and

$$a = \frac{1}{16} (f_{xxx} + f_{xxy} + g_{xxy} + g_{yyy}), \quad (5.3)$$

**using a weakly nonlinear asymptotic expansion for the solutions.** Note that this does not require you to understand (or even look) at the explanation in Guckenheimer and Holmes of how the criteria arises. What you are being asked to do is to provide an independent derivation of the criteria, using the techniques introduced in the lectures and notes — in particular, a two-times expansion of the solutions.

**Hint 1:** As explained in the lectures, whether the bifurcation is subcritical or supercritical depends on the nonlinearity being either stabilizing or de-stabilizing. Hence the issue can be decided by looking at the critical point $(0, 0)$ of (5.1), and checking what type of (nonlinear) spiral point it is — linearly it is a center, of course. **Be careful** with your choice of the slow time; it has to be scaled exactly so that it gives you the flexibility to eliminate the resonant terms at the lowest order that they show up.

**Hint 2:** The algebra involved is a lot simpler and elegant if you use complex notation. Namely, write the equation in the form

$$\frac{dz}{dt} - i \omega z = F(z, w), \quad (5.4)$$

where $z = x + i y, w = x - i y$ is the complex conjugate of $z$, and

$$F = f(x, y) + i g(x, y) = f \left( \frac{z + w}{2}, \frac{z - w}{2i} \right) + i g \left( \frac{z + w}{2}, \frac{z - w}{2i} \right).$$

Since $x = \frac{z + w}{2}$ and $y = \frac{z - w}{2i}$, the derivatives of $F$ with respect to $z$ and $w$ are easily related to the derivatives of $f$ and $g$ with respect to $x$ and $y$ — and vice-versa.

**THE END.**