1 Problem 6.1.13 - Strogatz (Draw a phase portrait).


Draw a phase portrait that has exactly three closed orbits and one fixed point.

2 Problem 6.5.06 - Strogatz (Epidemic model revisited).

Statement for problem 6.5.06.

(Epidemic model revisited.) In Exercise 3.7.6, you analyzed the Kerrmack-McKendrick model of an epidemic by reducing it to a first-order system. In this problem you will see how much easier
the analysis becomes in the phase-plane. As before, let \( x(t) \geq 0 \) denote the size of the healthy population and \( y(t) \geq 0 \) denote the size of the sick population. Then the model is

\[
\frac{dx}{dt} = -kxy, \quad \text{and} \quad \frac{dy}{dt} = kxy - \ell y, \tag{2.1}
\]

where \( k, \ell > 0 \) are constants. (The equation for \( z(t) \), the number of deaths, plays no role in the dynamics, so we omit it.)

a) Find and classify all the fixed points.

b) Sketch the nullclines and the vector field.

c) Find a conserved quantity for the system.

Hint: Form a differential equation for \( \frac{dy}{dx} \), separate variables and integrate both sides.

d) Plot the phase portrait. What happens as \( t \to \infty \)?

e) Let \( (x_0, y_0) \) be the initial condition. An epidemic is said to occur if \( y(t) \) increases initially. Under what conditions does an epidemic occur?

3 Problem 07.02.07 - Strogatz

(A system both potential and Hamiltonian).

Statement for problem 07.02.07.

Consider the system \( \dot{x} = y + 2xy = f \) and \( \dot{y} = x + x^2 - y^2 = g \).

a. Show that \( f_y = g_x \). Then exercise 7.2.5 (part a) implies that this is a gradient system.

b. Find \( V \).

c. Sketch the phase portrait.

Extra questions: (d) Show that the system is also Hamiltonian. (e) Show that the system has the complex form \( \dot{z} = i(\bar{z} + z^2) \), where \( z = x + iy \) and \( \bar{z} = x - iy \).

4 Problem 07.02.x1 - 385 extra problem (Area evolution).

Statement for problem 07.02.x1.

(Area evolution). Consider some arbitrary orbit, \( \Gamma \), for the phase plane system

\[
\frac{d\vec{r}}{dt} = \vec{F}(\vec{r}) \quad \text{where} \quad \vec{r} = (x, y)^T, \quad \vec{F} = (f(x, y), g(x, y))^T, \tag{4.1}
\]

and \( \vec{F} \) has continuous partial derivatives up to (at least) second order. That is: \( \Gamma \) is the curve in the plane given by some solution \( \vec{r} = \vec{r}_\gamma(t) \) to (4.1). Then
A. Let $\Omega = \Omega(t)$ be an “infinitesimal” region that is being advected, along $\Gamma$, by the flow given by (4.1). For example:

A1. Let $\Omega(0)$ be a disk of “infinitesimal” radius $dr$, centered at $\vec{r}_\gamma(0)$.
A2. For every point $\vec{r}_p^0 \in \Omega(0)$, let $\vec{r} = \vec{r}_p(t)$ be the solution to (4.1) defined by the initial data $\vec{r}_p(0) = \vec{r}_p^0$.
A3. Then, at any time $t_*$, the set $\Omega(t_*)$ is given by all the points $\vec{r}_p(t_*)$, where $\vec{r}_p^0$ runs over all the points in $\Omega(0)$.

Note that $\Omega(0)$ need not be a disk. Any infinitesimal region containing $\vec{r}_\gamma(0)$ will do. All we need is that the notion of area applies to it — see item B.

B. Let $A = A(t)$ be the area of $\Omega(t)$.

Find a differential equation for the time evolution of $A$. Notice that the equation that you will find is trivially extended to higher dimensions — e.g. to characterize the evolution of the volume in a 3-D phase space.

Hints.

h1. First, introduce the vector $\delta \vec{r} = \delta \vec{r}(t) = \vec{r}_p - \vec{r}_\gamma$ for every point in $\Omega(t)$. This vector characterizes the evolution of the “shape” of $\Omega$ as the set moves along $\Gamma$. In order to calculate how $A(t)$ evolves, you only need to know how the $\delta \vec{r}$ vectors evolve.

h2. For every vector $\delta \vec{r}$, write an equation giving $\delta \vec{r}(t + dt)$ in terms of $\delta \vec{r}(t)$ and the partial derivatives of $\vec{F}$ along $\Gamma$. Since you are dealing with infinitesimal terms, you can neglect higher order terms, so as to obtain a relationship from $\delta \vec{r}(t)$ to $\delta \vec{r}(t + dt)$ given by a linear transformation. Make sure that this linear transformation correctly includes the $O(dt)$ terms, which you will need to calculate time derivatives.

h3. From the transformation in item h2 derive a relationship between $A(t + dt)$ and $A(t)$ — use the fact that, for linear transformations, areas are related by the absolute value of the determinant. Notice that you need to calculate the determinant only up to $O(dt)$.

h4. Use the result in item h3 to calculate the time derivative of $A$, and thus obtain a differential equation for it.

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5  Problem 07.03.11 - Strogatz (Cycle graphs).

Statement for problem 07.03.11.

(Cycle graphs). Suppose $\dot{x} = f(x)$ is a smooth vector field on $\mathbb{R}^2$. An improved\(^1\) version of the Poincaré–Bendixon theorem states that: if a trajectory is trapped in a compact region, then it must approach a fixed point, a limit cycle, or something exotic called a cycle graph (an invariant set made of a finite number of fixed points, connected by a finite number of trajectories, all oriented either clockwise or counterclockwise). Cycle graphs are rare in practice;\(^2\) below is a contrived but simple example.

(a) Plot the phase portrait for the system

$$
\dot{r} = r (1 - r^2) \left( r^2 \sin^2(\theta) + (r^2 \cos^2(\theta) - 1)^2 \right) \quad \text{and} \quad \dot{\theta} = r^2 \sin^2(\theta) + (r^2 \cos^2(\theta) - 1)^2, \quad (5.1)
$$

where $r$ and $\theta$ are polar coordinates. **Hint:** Note the common factor in the two equations; examine where it vanishes.

(b) Sketch $x$ versus $t$ for a trajectory starting away from the unit circle. What happens as $t \to \infty$. **Hint:** plot $x$ versus $\log(t)$.

6  Problem 07.05.06 - Strogatz (Biased van der Pol).

Statement for problem 07.05.06.

(Biased van der Pol). Suppose the van der Pol oscillator is biased by a constant force:

$$
\frac{d^2 x}{dt^2} + \mu (x^2 - 1) \frac{dx}{dt} + x = a, \quad (6.1)
$$

where $a$ can be positive, negative, or zero. (Assume $\mu > 0$ as usual.)

a) Find and classify all the fixed points.

b) Plot the nullclines in the Liénard plane. Show that if they intersect on the middle branch of the cubic nullcline, the corresponding fixed point is unstable.

c) For $\mu \gg 1$, show that the system has a stable limit cycle if and only if $|a| < a_c$, where $a_c$ is to be determined. (**Hint:** Use the Liénard plane.)

d) Sketch the phase portrait for $a$ slightly greater than $a_c$. Show that the system is excitable — it has a globally attracting fixed point, but some (small, but not infinitesimal) disturbances can send the system on a long excursion through phase space before returning to the fixed point; compare with Exercise 04.05.03.

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\(^1\)This improved version is the one given at the lectures.

\(^2\)Not so rare, as shown at the lectures. Homoclinic connections are cycle graphs.

7 Problem 07.06.02 - Strogatz
(Calibrating regular perturbation theory).

Statement for problem 07.06.02.

(Calibrating regular perturbation theory). Consider the initial value problem $\ddot{x} + x + \epsilon x = 0$, with $x(0) = 1$, $\dot{x}(0) = 0$ and $0 < \epsilon \ll 1$.

a. Obtain the exact solution to the problem.

b. Using regular perturbation theory, find $x_0$, $x_1$, and $x_2$ in the series expansion
$$x(t, \epsilon) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + O(\epsilon^3).$$

c. Does the perturbation solution contain secular terms? Did you expect to see any? Why?

8 Problem 07.06.14 - Strogatz
(Computer test of two timing).

Statement for problem 07.06.14.

(Computer test of two timing). Consider the equation
$$\frac{d^2x}{dt^2} + \epsilon \left(\frac{dx}{dt}\right)^3 + x = 0,$$ (8.1)
where $0 < \epsilon \ll 1$.

a) Derive the averaged equations.

b) Given the initial conditions $x(0) = a$ and $\dot{x}(0) = 0$, solve the averaged equations and thereby find an approximate formula for $x = x(t, \epsilon)$.

c) Solve equation (8.1) numerically for $a = 1$, $\epsilon = 2$, $0 \leq t \leq 50$. Plot the results in the same graph as your part (b) answer. Note the impressive agreement, even though $\epsilon$ is not small! Challenge: can you explain why the agreement is so good?

THE END.