1 Separable CTRW (Continuous Random Walk)
Consider the sum of random variables $x_n$ with random waiting time $\tau_n$, where $x_n$ and $\tau_n$ are independent variables:

$$X(t) = \sum_{n=1}^{N(t)} x_n$$  \hspace{1cm} (1)

Here, the upper limit of sum $N(t)$ is a random function of continuous time.

We define:

$$\psi(t) = \text{PDF for } \tau_n (\text{IID})$$ \hspace{1cm} (2)

$$p(x) = \text{PDF for } x_n (\text{IID})$$

$$P(x,t) = \text{PDF for } X(t)$$

Recall that the Montroll-Weiss equation is

$$\tilde{\hat{p}}(k, s) = \left(1 - \frac{\psi(s)}{s}\right) \frac{1}{1 - \psi(s) \hat{p}(k)}$$ \hspace{1cm} (3)

As $k \to 0$, $s \to 0$, one can get moments of $X(t)$. We seek what kind of continuum equations for $p(x,t)$ are. Note that in this lecture, $< x > = 0$ by assumption, or in the other word, there is no drift.

2 Normal Diffusion

Now we consider the continuum limit of the continuous time random walk with normal diffusive scaling when CLT (central limit theory) holds. We assume that $< \tau >= \bar{\tau} < \infty$, $\sigma^2 < \infty$, and $\langle \Delta x \rangle = 0$, define $z(t) = \frac{x(t)}{\sigma \sqrt{N(t)}}$, then $\phi(z) = \frac{\exp(-z^2/2)}{\sqrt{2\pi}}$, where $N(t) = t/\bar{\tau}$.
The walker is assumed to have a finite mean waiting time, so the waiting-time distribution satisfies
\[ \psi(t) = o(t^{-2}), \]
and thus its Laplace transform will have a small \( s \)-expansion governed by
\[ \tilde{\psi}(s) \sim 1 - \bar{\tau} s, \ s \to 0, \]
and
\[ \hat{p}(k) \sim 1 - \frac{\sigma^2 k^2}{2}, \ k \to 0. \]

Substituting into Eq. (3), we have the long-time limit
\[ \tilde{\hat{p}}(k, s) \sim \frac{\bar{\tau}}{\bar{\tau} s + \frac{\sigma^2 k^2}{2} + \cdots} \sim \frac{1}{s + D k^2}, \]
where
\[ D = \frac{\sigma^2}{\bar{\tau}}. \]

The definition of Laplace transform is
\[ \tilde{\hat{p}}(k, s) = \int_0^\infty e^{-st} \hat{p}(k, t) dt. \]
Inverting Laplace Transform leads to
\[ \hat{p}(k, t) \sim e^{-Dk^2 t} = e^{-t/\bar{t}(k)}, \]
where \( \bar{t}(k) = \frac{1}{Dk^2} \), and is the exponential relaxation time for Fourier mode \( k \). Note that large \( k \) decays fast. As a result, \( p(x, t) \) approaches the solution of the normal diffusion equation,
\[ p(x, t) \sim \frac{e^{-x^2/4Dt}}{\sqrt{4\pi Dt}} \]
as \( t \to \infty \) and \( x = O(\sqrt{t}) \). (This is again the central limit theorem for CTRW.) Since the equation is linear, the same continuum limit holds for any initial condition of the CTRW.

Note
\[ z(t) = \frac{X(t)}{\sqrt{2Dt}} = \frac{X}{\sqrt{2t/\bar{\tau}}} = \frac{X(t)}{N(t)}. \]

To compare \( \hat{p}(k, t) \) and \( P(x, t) \): 1) \( \hat{p}(k, t) \) satisfies ODE
\[ \frac{\partial \hat{p}}{\partial t} = -\frac{\hat{p}}{\bar{t}(k)} \text{ with initial condition } \hat{p}(k, 0) = 1; \]
2) \( P(x, t) \) satisfies PDE
\[ \frac{\partial P}{\partial t} = D \frac{\partial^2 P}{\partial x^2}, \ P(x, 0) = \delta(x). \]

### 3 Super Diffusion

Assume \( \bar{\tau} \) is finite \( < \infty \), but \( \sigma^2 = \infty \) and symmetric \( p(-x) = p(x) \) (This is Levy flight). For example: \( p(k) = e^{-a|k|^\alpha} \ (0 < \alpha < 2), \ p(x) = \ell_{\alpha,a}(x). \)

Consider
\[ \tilde{\psi}(s) \sim 1 - \bar{\tau} s, \ s \to 0 \]
\[ \hat{p}(k) \sim 1 - B|k|^\alpha, \ k \to 0 \]

Using Eq.(3), we obtain
\[ \tilde{\hat{p}}(k, s) \sim \frac{\bar{\tau}}{\tau s + B|k|^\alpha + \ldots} \]

Invert Laplace Transform
\[ \hat{p}(k, t) \sim e^{-\frac{B}{\bar{\tau}}|k|^\alpha t} = e^{-t/\bar{\tau}|k|^\alpha} \]

To summarize, we still have exponential relaxation of Fourier modes in time, but now
\[ t(k) = \bar{\tau} \frac{B}{\bar{\tau}} |k|^\alpha, \alpha < 2. \]

So large \( k \) (or small wavelength) features decay more slowly compared to normal diffusion, but small \( k \) decay faster. Still
\[ \frac{\partial \hat{p}}{\partial t} = -\hat{p} \quad \text{ODE} \]
\[ \frac{\partial \hat{p}}{\partial t} = -\frac{B}{\bar{\tau}} |k|^\alpha \hat{p} \]

Let \( \kappa(t) = k\left(\frac{Bt}{\bar{\tau}}\right)^{\frac{1}{\alpha}} \leftrightarrow z = \frac{x}{(\frac{Bt}{\bar{\tau}})^{\frac{1}{\alpha}}} \), \( \hat{p} \sim e^{-|k(t)|\alpha} \), then
\[ P(x, t) \sim \left(\frac{\bar{\tau}}{Bt}\right)^{\frac{1}{\alpha}} \ell_{\alpha, 1} \left(\left(\frac{\bar{\tau}}{B}\right)^{\frac{1}{\alpha}} \frac{x}{t^{\frac{1}{\alpha}}} \right) \]

with scales like \( t^\nu \), where \( \nu = 1/\alpha > \frac{1}{2} \), the supper diffusion.

Note \( P(x, t) \) satisfies a fractional diffusion equation
\[ \frac{\partial P}{\partial t} = (\frac{B}{\bar{\tau}})^{\alpha} P \]

where \( \nabla^\alpha \) is the Riese fractional derivative which can be defined by:
\[ |\nabla|^\alpha f(k) = -|k|^\alpha \hat{f}(k) \]
\[ |\nabla|^\alpha f(x) = \int_{-\infty}^{\infty} e^{ikx}(-|k|^\alpha) \left( \int_{-\infty}^{\infty} e^{-ikx'} f(x') dx' \right) \frac{dk}{2\pi} \]
\[ = \int \int f(x') e^{ik(x-x')}|k|^\alpha dx' \frac{dk}{2\pi} = (f * \delta_\alpha)(x) \]

where
\[ \delta_\alpha = -\int_{-\infty}^{\infty} e^{ikx}|k|^\alpha \frac{dk}{2\pi} \]

When \( \alpha = 0 \), this is \( \delta(x) = \int e^{ikx} \frac{dk}{2\pi} \) and \( \delta(x) \) is localized. This function \( \delta_\alpha(x) \) is not localized in \( x \).

If \( \alpha \) is integer, \( |k|^\alpha = k^n \), \( \delta_n(x) = \frac{d^n}{dx^n} \delta(x) \), then \( |\nabla|^\alpha f \to \frac{d^n f}{dx^n} \).

Hence, boundary conditions for supper diffusion are subtle (fat tails in steps).
4 Subdiffusion

4.1 Mittag-Leffler Power-Law Decay of Fourier Modes

Consider symmetric (\( p(x) = p(-x) \)), anomalous subdiffusion with an infinite the mean waiting time (\( \langle \tau \rangle = \infty \)) but finite \( \sigma^2 \) for which the waiting-time distribution satisfies

\[
\psi(t) \sim \left( \frac{\tau_0}{\tau} \right)^{1+\gamma},
\]

and

\[
N(t) \sim t^\gamma,
\]

where \( 0 < \gamma < 1 \) or equivalently \( \tilde{\psi} \) has the following small-\( s \) expansion of its Laplace transform,

\[
\tilde{\psi}(s) \sim 1 - (\tau_0 s)^\gamma, \quad s \to 0.
\]

As \( k \to 0 \),

\[
\tilde{p}(k) \sim 1 - \frac{\sigma^2 k^2}{2}
\]

Thus, we have

\[
\tilde{p}(k,s) \sim \left( \frac{\tau_0 s} s \right)^\gamma \frac{1}{(\tau_0 s)^\gamma + \frac{\sigma^2 k^2}{2} + \ldots}.
\]

The factor in front of (4) is not a constant, and in fact is a singularity, as \( \gamma - 1 < 0 \). This crucial term, which is negligible in the case of normal diffusion, represents walks that have not moved yet.

We can rewrite (4) as

\[
\tilde{p}(k,s) \sim \frac{1}{s} \left( \frac{1}{1 + (\tilde{\tau}(k)s)^\gamma} \right),
\]

where

\[
\tilde{\tau}(k)^{-\gamma} = \tau_0^{-\gamma} \frac{\sigma^2 k^2}{2}.
\]

Or,

\[
\tilde{\tau}(k) = \frac{\tau_0}{k^{2/\gamma}} \left( \frac{2}{\sigma^2} \right)^{\frac{1}{2}} \propto \frac{1}{k^{2/\gamma}}
\]

Inverting Laplace transform gives

\[
\hat{p}(k,t) = E_{\gamma} \left( -(t/E(k))^\gamma \right)
\]

where \( E_{\gamma}(z) \) is Mittg-Leffler function, and \( E_{\gamma}(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(1+\gamma n)} \).

Note

\[
E_1(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(1+n)} = \sum_{n=0}^\infty \frac{z^n}{n!} = e^z,
\]

So we recover \( \hat{p}(k,t) = e^{-t/\tilde{\tau}(k)} \) for \( \gamma = 1 \).

\[
E_{1/2}(z) = e^{z^2} \text{erfc}(-z),
\]

Note
where erfc \( x \) is the complementary error function \( \text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-x^2} \, dx \), \( \text{erfc}(z) = 1 - \text{erf}(z) \).

In this case,

\[
\hat{p}(k, t) = e^{t/\bar{\ell}(k)} \text{erfc}(\sqrt{t/\bar{\ell}(k)})
\]

Asymptotics:

\[
\hat{p}(k, t) = E_\gamma(-t/\bar{\ell}(k)\gamma)
\]

For the non-exponential cases \( 0 < \gamma < 1 \), the asymptotic expansions of the Mittag-Leffler functions are

\[
E_\gamma(- (t/\bar{\ell}(k))\gamma) \sim \begin{cases} 
\exp\left(-\frac{((t/\bar{\ell})(k)\gamma)}{1+(t\bar{\ell}(k))^{-\gamma}}\right), & t \to 0 \\
\frac{1}{\Gamma(1+\gamma)} \left( \frac{\bar{\ell}(k)}{t} \right)^{\gamma}, & t \to \infty, \end{cases}
\]

so we have stretched-exponential decay at short times and power-law decay at long times.

Now what is the continuum relaxation equation?

\[
\frac{\partial \hat{f}}{\partial t}(s) = s \hat{f}(s) - f(0) = \int_0^\infty e^{-st} \frac{df}{dt}(t) \, dt
\]

\[
\frac{\partial \hat{p}}{\partial t}(k, s) = s \hat{p}(k, s) - \hat{p}(k, 0)
\]

At the long time limit in the central region

\[
\frac{\partial \hat{p}}{\partial t}(k, s) \sim \frac{1}{1 + (\bar{\ell}(k)s)^{-\gamma}} - 1
\]

\[
= \frac{1}{(ts)^{-\gamma}} - 1
\]

\[
= -\bar{\ell}(k)^{-\gamma} s^{1-\gamma} \hat{p}(k, s)
\]

Here, \( \bar{\ell}(k)^{-\gamma} = \frac{\sigma^2}{2 D_0} k^2 = D_\gamma^2 = -D_\gamma k^2 D^{1-\gamma}_t P(k, s) \).

For \( \hat{p} \), it satisfies equation

\[
\frac{\partial \hat{p}}{\partial t} = -D_\gamma k^2 \hat{D}_t^{1-\gamma} \hat{p}
\]

where \( \hat{D}_t^\beta \) is the Riemann-Liouville fractional derivative. So \( p(x, t) \) satisfies

\[
\frac{\partial p}{\partial t} = D_\gamma \left( \hat{D}_t^{1-\gamma} \right) \nabla^2 p.
\]

This is a fractional diffusion equation. For subdiffusion, boundary conditions are easy, but initial condition is subtle. Besides, \( \hat{D}_t^{1-\gamma} \) is nonlocal in time, depending on the history.