18.305 Fall 2011, Solutions to HW 9

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1 Problem 1

\[ J_p(x) = \frac{x^p}{2^p \sqrt{\pi} \Gamma(p + 1/2)} \int_{-1}^{1} e^{-ix\rho(1 - \rho^2)^{p-1/2}} \, d\rho \]

\[ = \frac{x^p}{2^p \sqrt{\pi} \Gamma(p + 1/2)} \int_{-1}^{1} \sum_{n=0}^{\infty} \frac{(-ix\rho)^n}{n!} (1 - \rho^2)^{p-1/2} \, d\rho \]

\[ = \frac{x^p}{2^p \sqrt{\pi} \Gamma(p + 1/2)} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \int_{0}^{1} t^{n-1/2} (1 - t)^{p-1/2} \, dt \]

\[ = \frac{x^p}{2^p \sqrt{\pi} \Gamma(p + 1/2)} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \frac{\Gamma(n + 1/2) \Gamma(n + p + 1)}{\Gamma(n + p + 1)} \]

\[ = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(n + p + 1)} \left( \frac{x}{2} \right)^{2n + p} \]

using \( t = \rho^2 \) and the tip given: \( \Gamma(2n + 1) = \frac{2^{2n} \Gamma(n + 1/2) \Gamma(n + 1)}{\sqrt{\pi}} \).

2 Chapter 9, Problem 4

We are asked to solve approximately

\[ \epsilon y'' + x(1 + x) y' + \frac{y}{2} = 0, \quad 0 < x < 1, \quad y(0) = 1, \quad y(1) = 2. \]  

We have \( a(x) = x(1 + x) > 0 \) so the boundary layer is at \( x = 0 \), where we have a turning point. In the case of a turning point, we know we have

\[ y'_\text{out} = -\frac{y_{\text{out}}}{2x(1 + x)} \Rightarrow y_{\text{out}} = A \sqrt{\frac{1 + x}{x}}. \]

We use the boundary condition at \( x = 1 \) to determine the constant \( A \) (since the rapidly varying function is negligible away from the boundary layer):

\[ y(1) = 2 \Rightarrow y_{\text{out}}(1) = A \sqrt{2} = 2 \Rightarrow A = \sqrt{2} \]
so that
\[ y_{\text{out}} = \sqrt{\frac{2(1 + x)}{x}} \]

Now we find the rapidly varying function \( y_{\text{in}} \), which is important near the boundary layer and turning point, at \( x = 0 \). We may use the solution the book provides, using the parabolic cylinder function:
\[ y_{\text{in}}(x) = e^{-\frac{x^2}{4}} \left( BD_{-1/2}(x/\sqrt{\epsilon}) + CD_{-1/2}(-x/\sqrt{\epsilon}) \right). \]

(If you are following the book, here we used \( \alpha = 1 \) and \( v = -1/2 \).) Now we need to determine the two constants \( B \) and \( C \). For this we need to satisfy the boundary condition at \( x = 0 \), but we also need our two solutions \( y_{\text{out}} \) and \( y_{\text{in}} \) to match away from the boundary layer. To match them, we need the asymptotic behavior of the \( D_{-1/2}(X) \). From what is given in the book, we see that \( D_{-1/2}(X) \approx \sqrt{2} |X|^{-1/2} e^{X^2/4} \) as \( X \rightarrow -\infty \) whereas \( D_{-1/2}(X) \approx X^{-1/2} e^{-X^2/4} \) as \( X \rightarrow \infty \). Note that here we have \( X = \pm x/\epsilon \) so for \( \sqrt{\epsilon} << x << 1 \), away from the boundary layer, \( |X| \) is large, hence the above approximations are valid. Hence we match both solutions in \( \sqrt{\epsilon} << x << 1 \):
\[ y_{\text{out}} = \sqrt{\frac{2(1 + x)}{x}} \approx \sqrt{(2/x)} = C\sqrt{(2/x)}e^{1/4} \approx y_{\text{in}} \]
so that \( C = \epsilon^{-1/4} \). Now we may find \( B \):
\[ y_{\text{in}}(0) = 1 \Rightarrow B = \frac{1}{D_{-1/2}(0)} - C = \frac{\Gamma(3/4)2^{1/4}}{\sqrt{\pi}} - \epsilon^{-1/4}. \]

And we may finally combine those in a uniform solution, which is to say we modify slightly the part of \( y_{\text{in}} \) which does not vanish as \( x \rightarrow 1 \), so that it matches the behavior of \( y_{\text{out}} \):
\[ y_{\text{uniform}} = e^{-\frac{x^2}{4}} \left( BD_{-1/2}(x/\sqrt{\epsilon}) + CD_{-1/2}(-x/\sqrt{\epsilon})\sqrt{1+x} \right), \]
with \( B \) and \( C \) as above.

3 Chapter 9, Problem 5

This Boundary Value Problem has the same boundary-layer structure as the BVP in pp.383-384. Following the same approach, we obtain for \( x > 0 \):
\[ y_{\text{in}}^{(a)}(x) = 3e^{-2(1-x)/\epsilon}. \]

For \( x < 0 \), we have
\[ y_{\text{in}}^{(b)}(x) = 2e^{-2(x+1)/\epsilon} \]

and since the slowly varying solution is the small one, we may omit it here. Hence
\[ y_{\text{uniform}}(x) = 3e^{-2(1-x)/\epsilon} + 2e^{-2(x+1)/\epsilon}. \]