18.305 Fall 2011, Solutions to HW 7

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1 Problem 1

\[ I(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda x} e^{i x^4} dx = \lambda^{1/3} \int_{-\infty}^{\infty} e^{i \Lambda (t + t^4)} dt \]

using \( t = \lambda^{-1/3} x \) and \( \Lambda = \lambda^{4/3} \). So \( f(t) = t + t^4, f'(t) = 1 + 4t^3, f'' = 12t^2 \). We find the saddle points \( (f' = 0 \Rightarrow t^3 = -1/4) \) to be

\[ t_0 = 4^{-1/3} e^{i\pi/3}, t_1 = -4^{-1/3}, t_2 = 4^{-1/3} e^{-i\pi/3}. \]

Since \( t_1 \) is on the original contour, this is the one we shall use. One could compute the contributions of each point, using \( f(t) = t(1 + t^3) = t(3/4) \) to find that \( t_0 \) gives an exponentially small contribution, \( t_2 \) gives an exponentially growing contribution hence it is irrelevant, and \( t_1 \) gives an oscillatory contribution: this is larger than the one coming from \( t_0 \), so it will be the main contribution. Now we note that \( f(t_1) = -3/4^{4/3} \) and \( f''(t_1) = 3 \times 4^{1/3} \). Thus the integral is approximated by the usual formula:

\[ I(\lambda) = \lambda^{1/3} e^{i \Lambda f(t_1)} \sqrt{\frac{2\pi}{-i\Lambda f''(t_1)}} = \lambda^{-1/3} 2^{1/6} \sqrt{\frac{\pi}{3}} \exp \left[ -3i\lambda^{4/3}/4 \times 2^{2/3} + i\pi/4 \right]. \]

We use Mathematica to approximate the solution numerically.

![Graph showing integral, asymptotics, and solution](image-url)
2 Problem 2

\[ I(\lambda) = \int_{-1}^{1} e^{i\lambda x} e^{-1/(1-x^2)} dx = \int_{-1}^{1} e^{i\lambda x} e^{-1/(2(1-x))} e^{-1/(2(1+x))} dx \]

and we use \( t = \sqrt{\lambda} (x - 1) \) to obtain

\[ I(\lambda) = \lambda^{1/2} e^{i\lambda} \int_{-2\sqrt{\lambda}}^{0} e^{i\lambda^{1/2} t} e^{\lambda^{1/2} / (2t)} e^{-1/(2(t/\lambda^{1/2}+2))} dt \]

so that \( f(t) = t - i/(2t) \), \( f'(t) = 1 + i/(2t^2) = 0 \Rightarrow t_0 = e^{3i\pi/4} / \sqrt{2} \) and \( t_1 = e^{-i\pi/4} / \sqrt{2} \). But the contribution from \( t_1 \) will be exponentially large, so we consider \( t_0 \). Hence \( f(t_0) = t_0(1 - i/(2t_0^2)) = 2t_0 = \sqrt{2}e^{3i\pi/4} \) and \( f''(t_0) = \frac{i}{2}(-2) \frac{1}{t_0^3} = \frac{2}{t_0} = 2^{3/2}e^{-3i\pi/4} \), so that, after simplifying and using \( h(t) = e^{-1/(2(t/\lambda^{1/2}+2))} \Rightarrow h(t_0) \approx e^{-1/4} \), we obtain

\[ I(\lambda) = \Re \left[ 2^{3/4} \lambda^{-3/4} e^{-1/4 - \sqrt{\lambda}} \frac{\pi e^{-i\pi/3+8i\lambda - i\sqrt{\lambda}}}{\sqrt{\pi}} \right] = 2^{3/4} \lambda^{-3/4} e^{-1/4 - \sqrt{\lambda}} \sqrt{\pi} \cos(-\pi/8 + \lambda - \sqrt{\lambda}). \]

Using Matlab, we verify that the approximation works.