1 Chapter 8, Problem 3

1.1 Solutions to part (a)

\[ I(\lambda) = \int_{-1}^{1} e^{-\lambda t^3} dt \]

We have

\[ v(t) = t^3, \]

and

\[ h(t) = 1 \]

\( v(t) \) is minimum at the lower endpoint \( t = -1 \), and \( v(-1) = -1, v'(-1) = 3 \). Thus, we have

\[ I(\lambda) \approx \frac{e^\lambda}{3\lambda} \] (1)

1.2 Solutions to part (b)

\[ I(\lambda) = \int_{1}^{\infty} e^{-\lambda t^2} dt \]

We have

\[ v(t) = t^2, \]

and

\[ h(t) = 1 \]

\( v(t) \) is minimum at the lower endpoint \( t = 1 \), and \( v(1) = 1, v'(1) = 2 \). Thus, we have

\[ I(\lambda) \approx \frac{e^{-\lambda}}{2\lambda} \] (2)
1.3 Solutions to part (c)

\[ I(\lambda) = \int_{-2}^{1} \sin x e^{-\lambda x^2} \, dx \]

\[ = \int_{-1}^{1} \sin x e^{-\lambda x^2} \, dx + \int_{-1}^{1} \sin x e^{-\lambda x^2} \, dx \]

\[ = \int_{-2}^{-1} \sin x e^{-\lambda x^2} \, dx \]

The integral from -1 to 1 vanishes because the integrand is an odd function. We then have

\[ v(x) = x^2, \]

and

\[ h(x) = \sin x \]

\( v(x) \) is minimum at the upper endpoint \( x = -1 \), and \( v(-1) = 1, v'(-1) = 2x \), \( h(x) = \sin(-1) \).

Thus, we have

\[ I(\lambda) \approx -e^{-\lambda \sin(-1)} - \frac{e^{-\lambda}}{2\lambda} \sin 1 \]  

(3)

1.4 Solutions to part (d)

\[ I(\lambda) = \int_{-\pi}^{\pi} e^{-\lambda \sin(x)} \, dx \]

We have

\[ v(x) = \sin x, \]

and

\[ h(x) = 1 \]

\( v(x) \) is minimum at an interior point \( x = -\pi/2 \), and \( v(-\pi/2) = -1, v''(-\pi/2) = 1 \). Thus, we have

\[ I(\lambda) \approx e^{\lambda} \sqrt{\frac{2\pi}{\lambda}} \]  

(4)

1.5 Solutions to part (e)

\[ I(\lambda) = \int_{0}^{\infty} e^{-\lambda x} e^{-x^2} \, dx \]

We have

\[ v(x) = x, \]

and

\[ h(x) = e^{-x^2} \]

\( v(x) \) is minimum at the lower endpoint \( x = 0 \), and \( v(0) = 0, v'(0) = 1, h(0) = 1 \). Thus, we have

\[ I(\lambda) \approx \frac{1}{\lambda} \]  

(5)
1.6 Solutions to part (f)

\[ I(\lambda) = \int_{0}^{\lambda} e^{t^3} dt \]

Applying the change of variable

\[ t = \lambda x, \]

the integral becomes

\[ I(\lambda) = \lambda \int_{0}^{1} e^{\lambda x^3} dx, \]

where

\[ \Lambda = \lambda^3 \]

We see that the dominant contribution comes from a small neighborhood near \( x = 1 \). Expanding around that point,

\[ x^3 \approx 1 + 3(x - 1) \]

Thus,

\[ I(\lambda) = \lambda \int_{0}^{1} e^{\Lambda x^3} dx \]
\[ \approx \lambda e^\Lambda \int_{0}^{1} e^{3\Lambda(x-1)} dx \]
\[ = \lambda e^\Lambda \left[ \frac{e^{3\Lambda(x-1)}}{3\Lambda} \right]_{0}^{1} \]
\[ = \lambda e^\Lambda \left[ 1 - e^{-3\Lambda} \right] \]
\[ \approx \lambda \frac{e^\Lambda}{3\Lambda} \]

\[ I(\lambda) \approx \frac{e^{\lambda^3}}{3\lambda^2} \quad (6) \]

1.7 Solutions to part (g)

\[ I(\lambda) = \int_{0}^{\infty} e^{-\lambda(t+t^5)} dt \]

We have

\[ v(t) = t + t^5, \]

and

\[ h(t) = 1 \]

\( v(t) \) is minimum at the lower endpoint \( t = 0 \), and \( v(0) = 0, v'(0) = 1 \). Thus, we have

\[ I(\lambda) \approx \frac{1}{\lambda} \quad (7) \]
2  Chapter 8, Problem 5

\[ I(\lambda) = \int_{0}^{\infty} e^{-\lambda x - \frac{1}{x}} dx \]

As was pointed out in the hint, we have to take into account both factors. Let

\[ h(x) = x + \frac{1}{\lambda x} \]

The minimum of \( h(x) \) occurs at

\[ \frac{1}{\sqrt{\lambda}} \]

To simplify our calculations, we apply the change of variables

\[ x = \frac{1}{\sqrt{\lambda}} t. \]

Thus,

\[ I(\lambda) = \int_{0}^{\infty} e^{-\lambda x - \frac{1}{x}} dx \]

\[ = \frac{1}{\sqrt{\lambda}} \int_{0}^{\infty} e^{-\sqrt{\lambda} \left( t + \frac{1}{t} \right)} dt \]

Now \( h(t) = t + 1/t \) and \( v(t) = 1 \). The minimum of \( h(t) \) occurs at \( t = 1 \). Around \( t = 1 \), we can make the approximation

\[ h(t) = 2 + \frac{h''(t)}{2} (t - 1)^2 = 2 + (t - 1)^2. \]

Therefore,

\[ I(\lambda) = \frac{1}{\sqrt{\lambda}} \int_{0}^{\infty} e^{-\sqrt{\lambda} (2 + (t - 1)^2)} dt \]

\[ = \frac{e^{-2\sqrt{\lambda}}}{\sqrt{\lambda}} \int_{0}^{\infty} e^{-\sqrt{\lambda} (t - 1)^2} dt \]

\[ = e^{-2\sqrt{\lambda}} \sqrt{\pi} \]

\[ = \frac{e^{-2\sqrt{\lambda}} \sqrt{\pi}}{\lambda^{3/4}} \]

3  Chapter 8, Problem 7

3.1  Solutions to part (a)

\[ I(\lambda) = \int_{-1}^{1} e^{-\lambda t^2} dt \]
Since the dominant contribution to this integral is around the endpoint $-1$, we may change the upper limit from 1 to $\infty$.

$$I(\lambda) = \int_{-1}^{1} e^{-\lambda t^3} dt \approx \int_{-\infty}^{\infty} e^{-\lambda t^3} dt$$

$$s = t^3$$

$$x = s + 1$$

$$y = \lambda x$$

$3.2$ Solutions to part (b)

$$I(\lambda) = \int_{1}^{\infty} e^{-\lambda t^2} dt$$

$$s = t^2$$

$$x = s - 1$$

$$y = \lambda x$$

$3.3$ Solutions to part (d)

$$I(\lambda) = \int_{-\pi}^{\pi} e^{-\lambda \sin x} dx$$
Since the dominant contribution to the integral is around $x = -\pi/2$, we have

\[
I(\lambda) \approx 2 \int_{-\pi/2}^{\pi/2} e^{-\lambda \sin x} dx
\]

\[
t = \sin x
\]

\[
s = \frac{t}{1 - t^2}
\]

\[
\approx 2e^\lambda \int_{0}^{\infty} e^{-\lambda s} \frac{1}{\sqrt{2s \sqrt{1 - s/2}}} ds
\]

\[
y = \lambda s
\]

\[
y = \frac{\sqrt{2} e^\lambda}{\sqrt{\lambda}} \sum_{n=0}^{\infty} \frac{\Gamma(n + 1/2)}{\Gamma(1/2) 2^n \lambda^n n!} \int_{0}^{\infty} e^{-y y^{(n-1)/2}} dy
\]

\[
3.4 \text{ Solutions to part (e)}
\]

\[
I(\lambda) = \int_{0}^{\infty} e^{-\lambda x} e^{-x^2} dx
\]

\[
y = \lambda x
\]

\[
\approx \frac{1}{\lambda} \int_{0}^{\infty} e^{-y} e^{-(y/\lambda)^2} dy
\]

\[
= \frac{1}{\lambda} \sum_{n=0}^{\infty} \frac{(-1)^n}{n! (-\lambda)^{2n}} \int_{0}^{\infty} e^{-y y^{2n}} dy
\]

\[
= \frac{1}{\lambda} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(2n + 1)}{n! \lambda^{2n}}
\]

\[
= \frac{1}{\lambda} \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{n! \lambda^{2n}}
\]
3.5 Solutions to part (f)

\[ I(\lambda) = \int_0^\lambda e^{t^3} dt \]

\[ I(\lambda) = \int_0^\lambda \frac{e^{x+\lambda^3}}{3(x+\lambda^3)^{2/3}} dx \]

\[ = \frac{e^{\lambda^3}}{3\lambda^2} \int_{-\lambda}^0 \frac{e^x}{(1+x/\lambda^3)^{2/3}} dx \]

\[ \approx \frac{e^{\lambda^3}}{3\lambda^2} \sum_{n=0}^{\infty} \frac{\Gamma(n+2/3)(-1)^n}{\Gamma(2/3)\lambda^{3n}} \int_{-\infty}^0 e^x x^n dx \]

\[ \approx \frac{e^{\lambda^3}}{3\lambda^2} \sum_{n=0}^{\infty} \frac{\Gamma(n+2/3)\Gamma(2/3)}{\lambda^{3n}} \]

3.6 Solutions to part (g)

\[ I(\lambda) = \int_0^\infty e^{-\lambda(t+t^5)} dt \]

\[ I(\lambda) = \int_0^\infty \frac{e^{-x}e^{-\lambda(x/\lambda)^5}}{\lambda} dx \]

\[ = \frac{1}{\lambda} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\lambda^{4n}} \int_0^\infty e^{-x} x^{5n} dx \]

\[ = \frac{1}{\lambda} \sum_{n=0}^{\infty} \frac{(-1)^n (5n)!}{n!\lambda^{4n}} \]